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OKE
Hirsch

I. C)

HIRSCH'S GEOMETRY;

OR

A SEQUEL

TO

EUCLID.

TRANSLATED FROM THE GERMAN

BY

THE REV. J. A. ROSS, A.M.

EDITED

BY

J. M. F. WRIGHT, A.B.

AUTHOR OF SOLUTIONS OF THE CAMBRIDGE PROBLEMS, &c. &c.

LONDON:

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1827.

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INTRODUCTORY PREFACE

BY

THE EDITOR.

FROM the encouragement bestowed upon our former labours—the Translations of Hirsch's Integral Tables, and of his First and Second Treatises upon Algebra—we are induced to lay before the Public another of the works of that most indefatigable and useful Author. Hirsch's Geometry, like his other productions, is vastly superior in every respect to any thing of the kind extant in our own language, and consequently will prove acceptable to every English Mathematician who shall look into it. In this country, Mathematicians who have ventured beyond the Elements of Euclid have, almost without exception, confined their speculations to "Deductions from Euclid," as the phrase goes, not daring, it would seem, to extend the Theories of that Father of Geometry. But Hirsch, being restrained by no such high feelings for antiquity, handles the subject in its generality, and strikes out a series of propositions, at once connected, and founded on views the most practical and beneficial. In all his works Hirsch is as elegant as he is useful, and perhaps none demands that eulogium more irresistibly than his Geometry. It is, indeed, both calculated to form the taste and direct the judg-

ment of the Mathematicians above alluded to; and indeed of almost all Geometers, without exception—at least in Britain.

Having thus spoken of the general merits of the work, we leave it to the reader to bear us out in particulars. We feel confident that all perusers will arrive at the same conclusion —viz. that *for those who have read Euclid, Hirsch's Geometry is by far the most useful and elegant production to be met with.*

Once for all, we must here explain a few difficulties. In page 29, Ex. 1. &c. it appears that the German measures are different from ours, not only in magnitude but in principle. They denote a certain measure of *Length,* viz. 1^0, which means one measure of that kind; therefore, 325^0 means 325 of such lengths. Then the parts of these measures they subdivide into 10ths, 100ths, 1000ths, &c., or *decimally,* in such a way that

$$325^0.\ 7'.\ 9''.\ 6'''.\ 5^{IV}.\ \&c.$$

means

$$325{\cdot}7965,\ \&c.$$

of the lengths denoted by the *superior* ^{0}.

In the first example, p. 29, for instance,

$$\frac{325{\cdot}79 \times 67{\cdot}83}{2} = 11049{\cdot}16785$$

which result is so many squares of which the side is 1^0, and which is therefore represented symbolically by

$$11049{\cdot}16785\ \square^0.$$

This again, by similar notation, with respect to the decimal parts, is the same as

$$11049\ \square^0\ 16\ \square'\ 78\ \square''\ 50\ \square''',$$

for the squares evidently are formed by successive 100ths.

Care must be taken not to mistake these measures of length with angular measures having the same notation.

Thus in p. 32 we have for the area of a triangle

$$q = \frac{ab\ Sin.\ \alpha}{2}$$

and for a particular example in this same page

$$q = \frac{257'.\ 9'' \times 356'.\ 3'' \times Sin.\ 25^0.\ 13'}{2}$$

Here the angular minutes are different from the minutes of length, being 60th parts of a degree, whereas the minutes of lengths are only 10ths of the degree of length.

It may be supposed that error may hence arise; but that will not be the case, since all functions of angles, such as *Sin. Cos. Tan. Sec.* &c. are numbers perfectly *pure or abstract.* Thus $Sin.\ 45^0 = \frac{1}{\sqrt{2}}$, $Sin.\ 30^0 = \frac{1}{2}$, $Tan\ 45^0 = 1$, and are therefore quite independent of all denominations of quantities, whether of length, weight, or any other kind.

Sometimes our author expresses his lengths wholly in minutes, sometimes wholly in degrees, which, of course, he is at liberty to do: in short the only thing for the student to bear in mind is the above distinction between lengths and angles. The necessity of this is very apparent in the examples down, p. 96.

J. M. F. W.

London, Oct. 1, 1827.

THE AUTHOR'S PREFACE.

I HERE give the public the Continuation announced in my Collection of Algebraical Problems and Formulæ, viz. the First Part of the Geometrical Collection, under the supposition that the First Part, at least, will not be received unfavourably. Were it not for the kind, and to me highly flattering, opinion of some persons I greatly value, I could not, for want of a critical judgment, presume to make this supposition, and to entertain the hope, that my labours, though they may not have entirely answered their expectations, yet have not altogether disappointed them. If, by omitting all that which is not necessary to the connection of the subject, brevity is an important requisite of a good mathematical book, since, agreeably to its design, it ought only to contain the fundamental *principles* and chief rules of the science; so it is equally necessary for the beginner to possess, also, another book which may afford him an opportunity of applying practically the rules he has already learnt, and by exercising himself in many different ways, prepare himself the better for reading a more comprehensive work. A book of this kind must, as it were, be intermediate between the first principles and such works as can be comprehended only by the more advanced, in order that the chasm between both may be filled up Thus, profound treatises on single subjects

are not so much required, because by their means no beginner can be grounded in those Elementary studies which they generally pre-suppose. Elementary treatises, interrupted here and there by occasional remarks, having a decided reference to practical application, appeared to me more desirable; by these the way is prepared to the former, and the foundation is laid for the future progress of the beginner.

That the geometrical object, if I may so express myself, requires nothing less than a facility of calculation in practice, every person will readily grant. This object, by which the experienced Geometrician as speedily sees the proportions and relations of the different parts of a figure, as the experienced calculator perceives the proportions and relations between numbers, can only be atained by a difference in the perspicuity and variety in the treatment. Geometry and Trigonometry, or, as the latter ought more properly to be called, Goniometry, furnishes us with ample materials for this purpose; arrangement and connection are only requisite to render them complete. Diffuse and *difficult* trigonometrical calculations, which only try the patience and not the head of the reader, cannot be absolutely considered as a means of improving the young geometrician; they are only so in reference to a higher object, and therefore only allowable in this view. It is calculations of this kind which have generally brought upon the inordinate admirers of the ancients, the unjust imputation, that their calculus does not tend to the improvement of the understanding, and that it lowers the sublime science of Geometry merely to mechanical purposes. But the entire separation of the geometrical method, so called, from the algebraical, has a favourable influence on the mind, since it accustoms it only to a certain form of thinking.

How far the present Collection fulfils all these negative conditions as it were, and whether the materials here made use of answer the above end, I cannot presume to decide. It only remained for me, therefore, in the above observations, to give the point of view in which I wish this treatise, on the whole, to be judged.

Respecting the problems themselves, they are partly taken from pure, partly from practical, Geometry; and, in the latter all technical terms are avoided both in the mode of treatment and expression. As in the algebraical collection, I have here strictly adhered to the fundamental rule, of proceeding from easy and simple examples, to more difficult and complicated ones, always mindful of the class of readers for whom I wrote. The treatises which I have made use of are mostly quoted, except those which, to avoid the imputation of plagiarism, I beg to mention in this Preface, viz. Schultz's Pocket Companion of Geometry; Schwab's Collection of 30 Geometrical Problems, as an Appendix to Euclid's Data; T. Simpson's Select Exercises for young Proficients in the Mathematics; Klügel's Mathematical Dictionary; also Mayer's Practical Geometry, which is not quoted in every place where it has been made use of. What is original will be readily perceived by the proficient.

The examples, of which there is an abundance, with the exception of two or three of the easiest, are all new; they are only omitted in the miscellaneous problems, because I no longer considered them necessary. In many of them, the final results only, together with a short notice of the mode of treatment, are given; in others, which involve long and complicated calculations, the intermediate results are also

adduced. This was done in order to save room; whether I have acted wisely in this respect, I shall, I trust, learn in time. However, I cannot be accused of doing it for the sake of any personal advantage or convenience. Should some of my readers not possess the information required for these cases, they must consult the tutor, or some proficient in the subject.

This First Part of the Geometrical Collection contains, besides, only Planimetrical problems, or such as may be classed under them, the Stereometrical are reserved for the Second Part. Much has been left out here rather unwillingly; to this belongs partly the doctrine of polygons in circles, which subject, since Gauss found the formula for the septemdecagon, appears to have excited some interest. The only excuse which I can offer for this omission is, that I had nothing new to say on the subject; however, my readers shall in due time be compensated for this.

The annexed Trigonometrical Tables of Formulæ contain such formulæ only as are used in the book. In quoting the Elements of Geometry, I have throughout referred to Lorenzini's Translation of Euclid, which probably is in the possession of most of my readers; the reference to it is denoted by Euc.; thus Euc. III. 27, implies the 27th Proposition of the Third Book of the Elements.

Berlin, Jan. 15, 1805.

CORRECTIONS.

Page 4, *line* 31, for $\triangle$ *ACD*, read $\triangle$ *AbD*
16, Head line, for *Algebra*, read *Analysis*
17, *line* 25, for *from I*, read *from T*
18, 5, for *CD*, read *AD*
32, 6, for *triangles*, read *angles*
32, last but one, for *Algebraically*, read *Geometrically*
40, 15, for *triangle*, read *quadrilateral*
47, 19, for *DE*, read *DF*
49, 18, for *BA*, read *BH*
56, 3, for $p = \infty$, read $x = \infty$
60, 2, for $e^2 = c^2 =$, read $e^2 = c^2 +$
62, 10, for *measured, from*, read *measured, in*
67, 20, for *radius*, read *diameter*
74, 2, for *Sin.* $(\gamma - \beta)$, read *Sin.* $(\gamma - \alpha)$
96, last but five, for *segment*, read *sector*
— last line, for *chord*, read *length*
97, *lines* 2, 3, 5, 10, 11, 12, 15, 19, 21, 26, for *segment*, read *sector*
98, 2, 14, 24, 26, for *segment*, read *sector*
99, 2, 10, 12, 15, 18, 20, 22, 24, 25, 27, 29, for *segment*, read *sector*
100, 3, 14, 17, 19, 21, 22, for *segment*, read *sector*
—— 14, 20, for *section*, read *segment*
103, *line* 17, for ϕ read *Sin.* ϕ
106, 28, 30, 33, for *segment*, read *sector*
107, 5, dele *curvilinear* $ACA' =$
110, 4, for a *Cos.* ϕ, read a *Cos.* α
—— 6, for *C*, read *D*

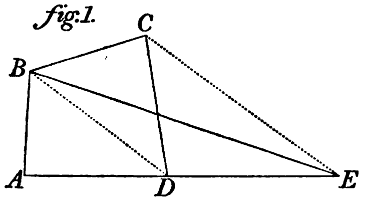
TRANSFORMATIO

I. TRANSFORMATION OF FIGURES.

SECTION I.

Problem. *To transform a given quadrilateral figure into a triangle, whose vertex is in a given angle of the figure, and whose base is in one of the sides of the figure.*

Construction. Let $ABCD$ (*fig.* 1 & 2), be the given quadrilateral; the *fig.* 1. has all its angles outwards, and the



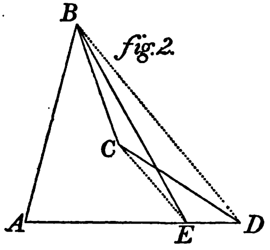


fig. 2 has one angle BCD inwards; let the vertex of the triangle, which is equal to it, fall in B.

1. Draw the diagonal BD (*fig.* 1 & 2), and, parallel to it, the line CE from C.

2. From E, where this line cuts AD (*fig.* 2), or its production (*fig.* 1), draw the line EB; then the $\triangle\, ABE$ = the quadrilateral figure $ABCD$.

Demonstration. Since $CE \parallel BD$, $\therefore$ (*fig.* 1 & 2) $\triangle BCD = \triangle BED$; consequently (*fig.* 1)

$$\triangle\, ABD + \triangle\, BCD = \triangle\, ABD + \triangle\, BED$$

or, the quadrilateral $ABCD = \triangle\, ABE$,

and (*fig.* 2)

$$\triangle ABD - \triangle BCD = \triangle ABD - \triangle BED$$

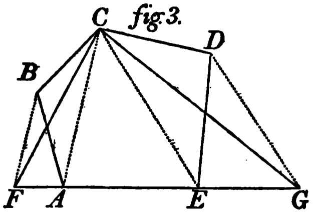
or, quadrilateral $ABCD = \triangle ABE$.

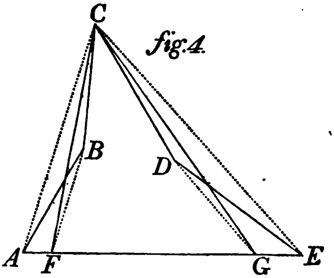
Corollary. If in the quadrilateral (*fig.* 1.), the side BC is parallel to its opposite side AD, then $BCED$ is a parallelogram; $\therefore BC = DE$, and consequently $AE = AD + DE = AD + BC$. In this case, the base AE of the triangle ABE is the sum of the two parallel sides. If $ABCD$ is a parallelogram, then $AD = BC = DE$, and $\therefore AE = 2\,AD$.

SECTION II.

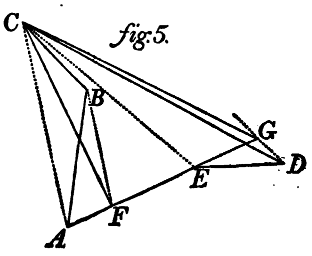
Prob. *Transform a given pentagon into a triangle, whose vertex is in a given angle of the pentagon, and whose base is in one of its sides.*

Const. Let $ABCDE$ (*figs.* 3, 4, 5) be the given





pentagon; let the vertex of the triangle which is equal to it fall in C.



1. From C draw the diagonals CA, CE.

2. From B draw BF parallel to CA, and from D draw DG parallel to CE.

3. To F and G, where these parallels cut AE or its production, draw the lines CF, CG; then CFG will be the triangle sought.

Demon. In all the three figures,

$$\triangle CBA = \triangle CFA, \ \triangle CDE = \triangle CGE.$$

1. (*fig.* 3)

$\triangle CAE + \triangle CBA + \triangle CDE = \triangle CAE + \triangle CFA + \triangle CGE$; or since $\triangle CAE + \triangle CBA + \triangle CDE =$ pentagon $ABCDE$, and $\triangle CAE + \triangle CFA + \triangle CGE = \triangle CFG$; $\therefore$ the pentagon $ABCDE = \triangle CFG$.

2. (*fig.* 4)

$\triangle CAE - \triangle CBA - \triangle CDE = \triangle CAE - \triangle CFA - \triangle CGE$; or since $\triangle CAE - \triangle CBA - \triangle CDE =$ pentagon $ABCDE$, and $\triangle CAE - \triangle CFA - \triangle CGE = \triangle CFG$; $\therefore$ the pentagon $ABCDE = \triangle CFG$.

3. (*fig.* 5)

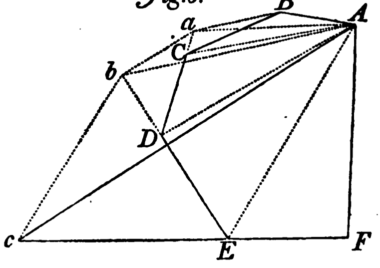
$\triangle CAE - \triangle CBA + \triangle CDE = \triangle CAE - \triangle CFA + \triangle CGE$; or since $\triangle CAE - \triangle CBA + \triangle CDE =$ pentagon $ABCDE$, and $\triangle CAE - \triangle CFA + \triangle CGE = \triangle CFG$; $\therefore$ the pentagon $ABCDE = \triangle CFG$.

SECTION III.

Prob. *To convert any given figure into a triangle, whose vertex shall be in a given angle of the figure, and whose base is in one of its sides.*

Const. Let $ABCDEF$ (*fig.* 6 & 7) be the given figure, in this case a hexagon, and A the angle in which the vertex of the given triangle is situated. For the sake of perspicuity, I shall enumerate the angles and sides of the figure from A, and call the first angle A, the second B, the third C, the fourth D, and so on; further,

fig.6.



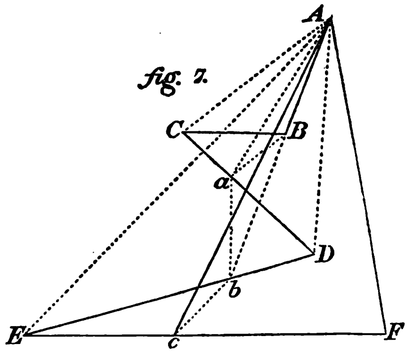


fig. 7.

AB the first side, *BC* the second, *CD* the third, *DE* the fourth, and so on. We shall then have the following general solution:

1. From *A* to all the angles of the figure, draw the diagonals *AC*, *AD*, *AE*, which, according to the order in which they stand here, call the first, second, third diagonal.

2. Then draw from the second angle *B* a line parallel to the first diagonal *AC*; from the point *a*, where this parallel meets the third side *CD* (*fig.* 7), or its production (*fig.* 6), draw a line *ab* parallel to the second diagonal *AD*, and from the point *b*, where this meets the fourth side *DE* (*fig.* 7), or its production (*fig.* 6), draw another line *bc* parallel to the third diagonal *AE*.

3. Continue this till there are no more diagonals, and from the last point of section of the parallels and sides, (in this case *c*), draw the line *cA*, then *AcF* is the required triangle, whose vertex is in *A*, and whose base is in the side *EF*.

DEMON.

1. (*fig.* 6). $\triangle\ ABC = \triangle\ AaC$ (because $Ba \parallel AC$); consequently

$$ACDEF + \triangle\ ABC = ACDEF + \triangle\ AaC;$$

or the hexagon $ABCDEF$ = pentagon $AaDEF$.

2. $\triangle AaD = \triangle ACD$ (because $ab \parallel AD$); consequently

$$ADEF + \triangle\ AaD = ADEF + \triangle\ AbD;$$

or the pentagon $AaDEF$ = quadrilateral $AbEF$.

3. $\triangle\ AbE = \triangle\ AcE$ (because $bc \parallel AE$); consequently

$$\triangle\ AEF + \triangle\ AbE = \triangle\ AEF + \triangle\ AcE;$$

or the quadrilateral $AbEF = \triangle\ AcF$.

1. (*fig.* 7). $\triangle ABC = \triangle AaC$ (because $Ba \parallel AC$); consequently

$$ACDEF - \triangle ABC = ACDEF - \triangle AaC;$$

or the hexagon $ABCDEF =$ the pentagon $AaDEF$.

2. $\triangle AaD = \triangle AbD$ (because $ab \parallel AD$); consequently

$$ADEF + \triangle AaD = ADEF + \triangle AbD;$$

or the pentagon $AaDEF =$ quadrilateral $ACEF$.

3. $\triangle AbE = \triangle AcE$ (because $bc \parallel AE$); consequently

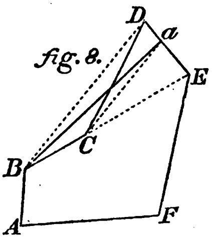
$$\triangle AEF - \triangle AbE = \triangle AEF - \triangle AcE,$$

or the quadrilateral $AbEF = \triangle AcF$.

We ∴ have for both figures, the hexagon $ABCDEF =$ pentagon $AaDEF =$ quadrilateral $AbEF = \triangle AcF$.

First Remark. Although the solution here given is only intended for a hexagon, yet it may easily be applied to every other figure. All depends upon the substitution of one triangle for another by means of parallel lines, in which we have only to take care, that one side of the triangle substituted, be in one side of the figure, or in its production, because by these means the number of its sides will be diminished. Moreover, it is not absolutely necessary actually to draw the parallels; it is only requisite, for instance, to note the points at which they cut its sides or their productions, because all depends upon the determination of these points of section.

Second Remark. A cursory inspection of the constructions and demonstrations in §§ I, II, and III, will show a certain agreement between the figures with angles tending outwards and inwards, by means of which the constructions and demonstrations may be transferred almost literally from one figure to the other. The difference consists merely in the signs $+$ and $-$, or in addition and subtraction.

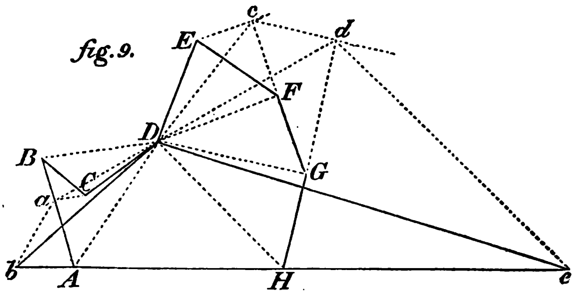


Cor. 1. In the figure $ABCDEF$ (*fig.* 8) it happens, that when we assume E as the vertex of the required triangle, the side BC of the figure is in a straight line with the diagonal EC. If we wish to proceed according to the above directions, we must through D draw a line parallel to EC, which, however, would never meet BC, as is required for the further construction. But this circumstance may be easily remedied, by taking away, previously to the application of the given rules, the angle BCD which tends inwards. Thus if

we draw the diagonal BD, and Ca parallel to it, then, when Ba is drawn, $\triangle CBa = \triangle CDa$, and $\therefore$ the figure $ABaEF =$ figure $ABCDEF$. This taking away of the angle which inclines inwards, may also be made use of with advantage, in the case where the point of section of the parallels and sides falls far without the figure, and in which the construction would consequently occupy too much room.

Cor. 2. If the vertex of the triangle is not only situated in a given angle of the figure, but the base of the triangle is in a certain side of the figure, as, for instance, when the octagon $ABCDEFGH$ is transformed (*fig.* 9) into

fig. 9.



a triangle, whose vertex is in D, and whose base is in AH, we can then proceed in the following way: Draw the lines DA, DH; then the figure is divided into three parts, viz. into the triangle DAH, the quadrilateral figure $DCBA$ on the left, and the pentagon $DEFGH$ on the right. Now transform, according to the given rules, the quadrilateral figure $DCBA$ into the triangle DaA; draw from a the line ab parallel to DA, and to the point b, where it meets the production AH, the line Db, then triangle $DbA =$ triangle $DaA =$ quadrilateral figure $DCBA$. In the same manner, transform the pentagon $DEFGH$ into the triangle DdH; draw the line de parallel to DH, and from the point, where it cuts the production AH, the line De; then the triangle $DeH =$ triangle $DdH =$ pentagon $DEFGH$. Therefore triangle $bDE =$ octagon $ABCDEFGH$; the vertex of the triangle is situated, as was required, in D, and the base is in the side AH. Further, to complete the proof, the lines Da, Dc are drawn.

SECTION IV.

PROB. *To transform any given figure into a triangle, whose vertex is in a certain point in one of the sides of the figure, or within it, and whose base is situated in a given side of the figure.*

SOLUTION. *First Case.* Let (*fig.* 10) *ABCDEF* be the

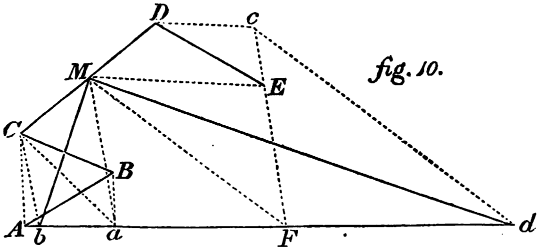


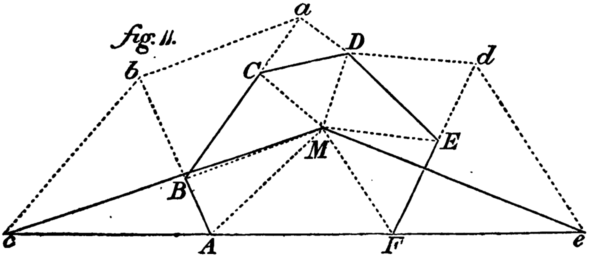
fig. 10.

hexagon which is to be transformed into a triangle; let its vertex be in the point *M* in the side *CD*, and the base in *AF*.

1. In the first place, by § III, Cor. 1. omit the angle *ABC* which inclines inwards; since the triangle *BCa* is substituted for the triangle *BAa*, and by these means the hexagon *ABCDEF* is transformed into the pentagon *aCDEF*: then draw the lines *Ma*, *MF*, and the pentagon *aCDEF* is divided into the triangle *MaF*, the quadrilateral figure *MDEF* on the right, and the triangle *MCa* on the left.

2. Transform the quadrilateral figure *MDEF* and the triangle *MCa* into the triangles *MdF*, *Mba*, so that the bases may be in *AF*; then *Mbd* = hexagon *ABCDEF*.

Second Case. Let *ABCDEF* (*fig.* 11) be the given

fig. 11.


figure; let the vertex of the required triangle be situated in the point M within the figure, and let the base be in AF.

1. From M to any angle of the figure, say D, draw the line MD, and draw the lines MA, MF, by which means the figure $ABCDEF$ is divided into the triangle MAF, and the figures $MDCBA$, $MDEF$.

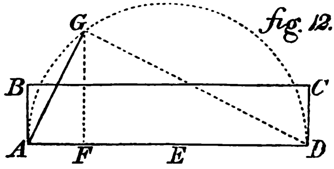
2. Then transform $MDCBA$ and $MDEF$ into the triangles McA, MeF, whose bases are in AF; then triangle cMe = figure $ABCDEF$.

SECTION V.

PROB. *To transform a given rectangle into a square.*

CONST. Let $ABCD$ (*fig.* 12) be the given rectangle, which is to be transformed into a square.

fig. 12.



1. Bisect the longest side AD in E.

2. With a radius $EA = ED$, describe upon AD the semicircle AGD.

3. From AD cut off a part AF equal to the least side AB of the rectangle.

4. From F draw the perpendicular FG.

5. From the point G, where this perpendicular meets the circle, draw the straight line GA; then this is a side of the square sought.

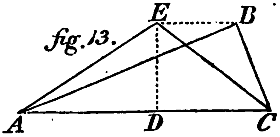
DEMON. Draw the line GD; then AGD is a right angle (*Euc.* III. 31), consequently AG is a mean proportional between AD and AF (*Euc.* VI. 8), ∴ (*Euc.* VI. 17) $AG^2 = AD \times AF = AD \times AB =$ the rectangle $ABCD$.

REMARK. Any figure may be converted into a triangle, and any triangle into a rectangle (*Euc.* 1. 42); further, any rectangle, as appears from the above problem, may be converted into a square; consequently also any figure into a square.

SECTION VI.

PROB. *To convert any given triangle into an isosceles one.*

CONST. Let *ABC* (*fig.* 13) be the given triangle, which is to be converted into an isosceles one.



1. Bisect the base *AC* in *D*, and from *D* draw the perpendicular *DE*.

2. From the vertex *B* of the given triangle, draw *BE* parallel to the base *AC*.

3. From the point *E*, where this parallel meets the perpendicular, draw the straight lines *EA*, *EC*; then *EAC* is the isosceles triangle sought.

The demonstration is very easily found.

SECTION VII.

PROB. *To convert a given isosceles triangle into an equilateral one.*

CONST. Let *ABC* (*fig.* 14) be the given isosceles triangle, which is to be converted into an equilateral one.

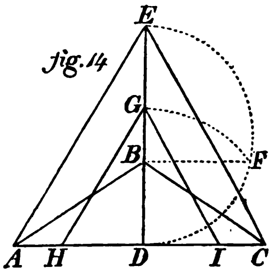
1. Upon the base *AC* of the given triangle, draw the equilateral triangle *AEC*, and through the vertices *E*, *B* of both the triangles, draw the straight line *EB*, which evidently is perpendicular to *AC*, and bisects the last line in *D*.

2. Upon *ED* describe the semicircle *EFD*, and from *B* draw the perpendicular *BF*, which meets the semicircle in *F*.

3. From D with the radius DF, describe an arc FG, which cuts the line DE in G.

4. From G draw the lines GH, GI, parallel to the sides of the equilateral triangle AEC; then HGI will be the equilateral triangle sought.

fig. 14



DEMON. Since $GH \parallel EA$, and $GI \parallel EC$, the $\angle\ GHI = \angle\ EAC$, and $\angle\ GIH = \angle\ ECA$; consequently $\triangle\ AEC \backsim$ * $\triangle\ HGI$, and $\therefore$ the $\triangle\ HGI$ is equilateral.

Suppose the line DF drawn, then DF, consequently also DG is the mean proportional between DE and DB, and $\therefore$

$$DE : DG = DG : DB;$$

or also

$$DE : DG = AD : HD, \text{ (because } \triangle\ EAD \backsim \triangle\ GHD);$$

consequently

$$DG : DB = AD : HD,$$

and $\therefore$ because the $\angle\ GDA$ is common,

$$\triangle\ GHD = \triangle\ BAD \ (Euc.\ \text{VI}.\ 15)$$

But $\triangle\ GHD = \triangle\ GID$, and $\triangle\ BAD = \triangle\ BCD$ consequently also $\triangle\ ABC = \triangle\ HGI$.

COROL. If BD be greater than ED, then the perpendicular BF does not meet the semicircle. In this case it will merely be necessary to describe the semicircle on BD, and from E to draw the perpendicular; then in this case the points H, I will not be situated in the line AC, but in its production.

REMARK. From this and the preceding §§, it appears, how any figure may be converted into an equilateral triangle; for it is only necessary first to convert the figure into a triangle, this triangle into an isosceles triangle, and the isosceles triangle again into an equilateral triangle.

* Wherever the symbol ($\backsim$) is used, it denotes "similar to."—*Translator.*

SECTION VIII.

Prob. *To describe a square which is equal to the sum of several given squares.*

Const. Let ab, cd, ef, gh, (*fig.* 15) be the sides of four squares: required to find a square which is equal to the sum of these four squares.

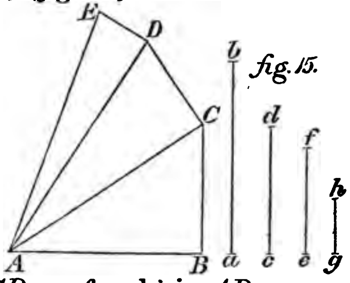


fig. 15.

1. Make $AB = ab$, and from B draw the perpendicular $BC = cd$, and join AC.

2. From the point C in AC draw the perpendicular $CD = ef$, and join AD.

3. From the point D in AD draw the perpendicular $DE = gh$, and join AE, then $AE^2 = ab^2 + cd^2 + ef^2 + gh^2$.

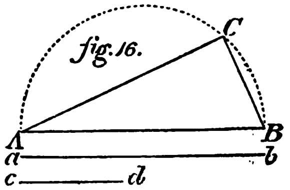
The demonstration is very easily deduced from *Euc.* I. 47.

Remark. Hence it appears how any number of squares may be converted into a single one. Since every figure may be converted into a square (§ V. Remark), consequently a square may always be found which is equal to the sum of several figures.

SECTION IX.

Prob. *To describe a square which is equal to the difference of two given squares.*

Const. Let ab, cd, (*fig.* 16) be the sides of two squares: required to find a square, which is equal to their difference.



1. Make $AB = ab$, and on AB describe a semicircle.

2. From B within the semi-

circle, draw the line $BC=cd$, and join AC; then AC is the side of the square sought.

The demonstration is deduced from *Euc.* III. 31, and I. 47.

SECTION X.

Prob. *Let several similar figures be given; construct a figure which is similar to each of them, and equal to their sum.*

Const. Let ab, cd, ef, gh, (*fig.* 15) be the homologous sides of four similar figures, to which, collectively, one similar figure is required to be made.

Upon the lines ab, cd, ef, gh, construct right angles, exactly as in § VIII, and by *Euc.* VI. 18, describe upon AE a figure, which is similar to the given ones; then this will be equal to the sum of the four figures.

The demonstration is founded on *Euc.* VI. 31, and is easily derived from it.

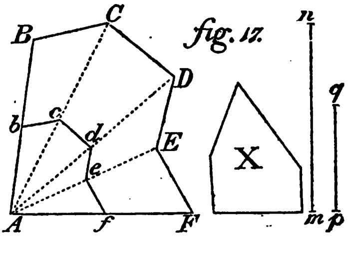
Remark. By the above method, we are enabled to convert several similar triangles, quadrilateral figures, pentagons, and so on, into a triangle, quadrilateral figure, pentagon, and so on, similar to them. Since the areas of circles are as the squares of their diameters, consequently there always may be found a circle, which is equal to the sum of several given circles. Thus, let ab, cd, ef, gh, be the diameters of four circles, then a circle, whose diameter is AE, is equal to the sum of these four circles. If we proceed with the diameters of two given circles, as we did in § IX with the sides of the two given squares, we shall find the diameter of a circle, which is equal to the difference of these two circles.

SECTION XI.

Prob. *To transform a given figure in such a way, that it may be similar to another figure.*

Const. Let X (*fig.* 17) be the given figure, and

ABCDEF the one to which it is to be similar.



1. Convert the figure *ABCDEF* into a square, and let its side be *mn*, so that $ABCDEF = mn^2$; convert also the figure X into a square, and let its side be *pq*, so that $X = pq^2$.

2. Take any side of the figure, say *AF*; to the three lines *mn*, *pq*, *AF*, find a fourth proportional (*Euc.* VI. 12), which cut off from *AF*; let *Af* be this fourth proportional, so that $mn : pq = AF : Af$.

3. Then draw the diagonals *AE*, *AD*, *AC*, and the lines *fe*, *ed*, *dc*, *cb*, parallel to the lines *FE*, *ED*, *DC*, *CB*; then *Abcdef* will be the required figure, viz. $=$ X and $\backsim$ *ABCDEF*.

Demon. It may be easily proved that $Abcdef \backsim ABCDEF$. Further, since by *Euc.* VI. 20.

$$ABCDEF : Abcdef = AF^2 : Af^2,$$

and by (2) $AF^2 : Af^2 = mn^2 : pq^2$,

then $ABCDEF : Abcdef = mn^2 : pq^2$;

But by (1) $ABCDEF = mn^2$,

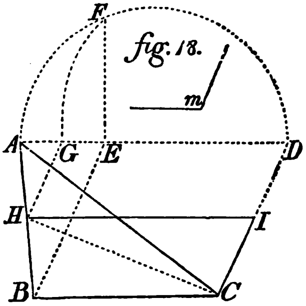
consequently also $Abcdef = pq^2 = X$.

SECTION XII.

Prob. *Upon the base of a given triangle, to describe a quadrilateral figure, equal to the given triangle, with two parallel sides, one of which is the base itself, and one of whose angles at the base is also one of the angles of the triangle, and the other angle is equal to a given angle.*

Const. Let *ABC* (*fig.* 18) be the given triangle, which

is required to be converted into a quadrilateral figure, one of whose sides is the base of the triangle, and another parallel to it; further, let the angle ABC of the triangle be also one of the angles of the quadrilateral figure, and the other angle at C equal to the given angle m.



1. On BC make the angle $BCD = m$.

2. From the vertex A of the triangle, draw AD parallel to BC, which meets CD in D, and from B draw the line BE parallel to CD, which meets AD in E.

3. Upon AD describe the semicircle AFD, and from E draw the perpendicular EF which meets the semicircle in F.

4. From D, with the radius DF, describe the arc FG, which meets AD in G.

5. From G draw the line GH parallel to DC, and from the point H, in which it meets AB, the line HI parallel to BC; then $BHIC$ is the quadrilateral figure sought.

Demon. Suppose DF drawn, then, by *Euc.* III. 31 and VI. 8, because also $DF = DG$,

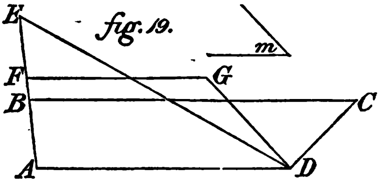
$$AD : DG = DG : DE,$$

consequently $AD - DG : DG - DE = DG : DE$,
or $AG : GE = HI : BC$; (because $DG = HI$, and $DE = BC$)
but $AG : GE = AH : HB$,
consequently $HI : BC = AH : HB$.
Further, $\left.\begin{array}{l} HI : BC = \triangle HCI : \triangle CHB \\ \text{and } AH : HB = \triangle ACH : \triangle HCB \end{array}\right\}$ *Euc.* VI. 1.
consequently $\triangle HCI : \triangle CHB = \triangle ACH : \triangle CHB$,
$\therefore \triangle HCI = \triangle ACH$,
and $\triangle HCI + \triangle HCB = \triangle ACH + \triangle HCB$;
or the quadrilateral figure $BHIC = \triangle ABC$.

SECTION XIII.

PROB. *To convert a given quadrilateral with two parallel sides, into another quadrilateral with two parallel sides, having one side and one of the angles adjacent to this side, common with the former, and whose angle at the other side is equal to a given one.*

CONST. The quadrilateral $ABCD$ (*fig.* 19), in which the two sides AD, BC are parallel, is to be converted into another which has the angle A and line AD in common, but the other angle at D equal to the given one m.



Convert the quadrilateral $ABCD$, by § I. into the triangle DAE, and this again, by § XII., into the quadrilateral $AFGD$, so that the angle $ADG = m$, then $AFGD$ is the quadrilateral sought.

II. DIVISION OF FIGURES BY ALGEBRA.

SECTION XIV.

AUXILIARY RULE.

PROB. *To divide a given straight line according to a given proportion.*

CONST. Let the line Ad (*fig.* 20) be given: divide this line in such a way, that the parts, in the order in which they follow each other, may be in the same proportion as the lines m, n, p are.

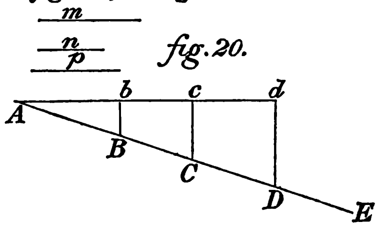


fig. 20.

1. Draw AE forming any angle with Ad, and from A towards E, draw the lines m, n, p, so that $AB = m$, $BC = n$, $CD = p$.

2. From the last point D draw the line Dd, and from B and C the lines Bb, Cc parallel to Dd, then b and c are the points of section of the line Ad; thus $Ab : bc : cd = m : n : p$.

The reason of this mode of treatment is easily seen.

COR. If, therefore, it is required to divide a straight line into a certain number of equal parts, it is only necessary upon AE to draw the same number of equal parts of arbitrary magnitudes, and then proceed according to 2.

SECTION XV.

Prob. *To divide a triangle from its vertex into a given number of equal parts.*

Const. Let ABC (*fig.* 21) be the given triangle, which is to be divided, say, into six equal parts: let A be the vertex, from which the lines of division are to be drawn.

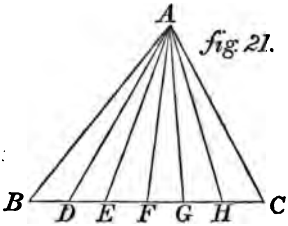


fig. 21.

1. Divide the side BC opposite the vertex A into six equal parts BD, DE, EF, FG, GH, HC.

2. From A to the points of division D, E, F, G, H, draw the lines AD, AE, AF, AG, AH; then the triangle ABC is divided into the six equal triangles ABD, ADE, AEF, AFG, AGH, AHC.

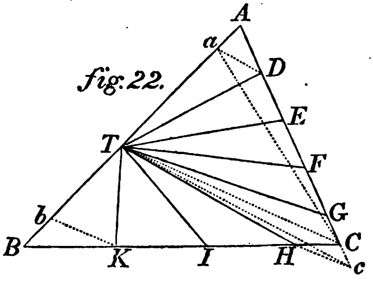
Cor. If it is required to divide the triangle ABC according to a given proportion, it will only be necessary to divide the line BC in this proportion (§ XIV), and from A to draw lines to these points of division.

SECTION XVI.

Prob. *From a given point in one of the sides of a triangle to divide it into a given number of equal parts.*

Const. Let ABC (*fig.* 22) be the given triangle, which is to be divided into eight parts; the lines of division are to be drawn from I.

fig. 22.



1. Make $Aa = Bb = \frac{1}{8}AB$, and from T draw the line TC to the vertex of the triangle.

2. From a and b draw the lines aD, bK, pa-

rallel to TC, which meet the sides AC, BC, in D and K.

3. From AC, from A towards C, cut off as many lines equal to CD as possible, (in this case four), and by these means determine the points E, F, G; from BC also cut off as many lines equal to BK as possible, (here three), and by these means determine the points I, H.

4. From T draw the lines TD, TE, TF, TG, TH, TI, TK, then ATD, DTE, ETF, FTG, $GTHC$, HTI, ITK, KTB, are the eight equal parts of the triangle ABC.

Demon. Draw Ca, then

$$\triangle ACB : \triangle ACa = AB : Aa = 8 : 1$$

$$\text{and } \therefore \triangle ACa = \tfrac{1}{8} \triangle ACB;$$

$$\text{but } \triangle ACa = ATD$$

because $\triangle AaD$ is common to both triangles, and $\triangle aTD = \triangle aCD$; consequently also

$$\triangle ATD = \triangle DTE = \triangle ETF = \triangle FTG = \tfrac{1}{8} \triangle ABC.$$

In like manner it may be proved, by drawing the line bC, that

$$\triangle BTK = \triangle KTI = \triangle ITH = \tfrac{1}{8} \triangle ABC.$$

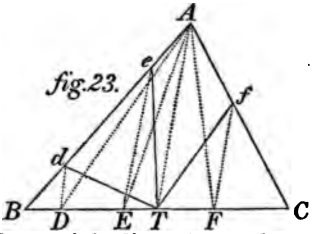
Since $\therefore$ the triangles taken together compose $\frac{7}{8}$ of the triangle ABC, consequently the quadrilateral $CGTH$ must in like manner be the eighth part of the triangle ABC.

Cor. 1 If after having cut off as many lines equal to AD as possible, we arrive at G, and wish to determine immediately the point H in the line BC, this may be very easily represented in the following simple manner. Produce AC, and from G make another line Gc equal to AD; then from c draw cH parallel to CT; then the point H is determined. For since $\triangle THC = \triangle TcC$ (because $cH \parallel CT$); then, when the $\triangle TCG$ is added, the quadrilateral $GTHC = \triangle GTc$; but $\triangle GTc = \triangle ATD = \frac{1}{8} \triangle ABC$; consequently also the quadrilateral $GTHC = \frac{1}{8} \triangle ABC$.

Cor. 2. If it is required from a given point in one of the

sides of a triangle, not to divide it into equal parts, but according to a different proportion, then it will be best to proceed in the following way:—

Let ABC (*fig.* 23) be the triangle to be divided, and T the point, from which the lines of division are to be drawn.



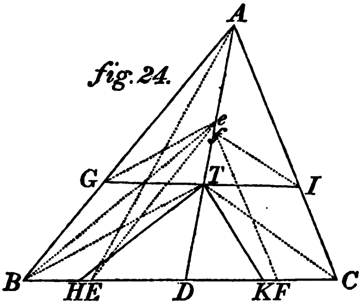
Divide the line BC, in which the point T is situated, according to the proportion given in § XIV, say in D, E, F; draw AT, and parallel to it the lines Dd, Ee, Ff, which meet the sides AB, AC, in d, e, f; from these points draw the straight lines dT, eT, fT, then BTd, dTe, $TeAf$, fTC, are the parts sought.

For if we draw the lines AD, AE, AF, the triangles ABD, ADE, AEF, AFC are as their bases BD, DE, EF, FC (*Euc.* VI. 1), and since these triangles, as may be easily proved, are respectively equal to the parts BTd, dTe, $TeAf$, fTC; consequently also these last are as the given lines.

SECTION XVII.

Prob. *To divide a triangle, from a point within it, into a given number of equal parts.*

Const. Let ABC (*fig.* 24) be the given triangle, which is to be divided into five equal parts; T the point from which the lines of division are to be drawn.



1. Take any side of the triangle, say BC, and make, when, as here, the triangle is to be divided into five parts, $BE = CF = \frac{1}{5}BC$, and draw the lines Ee, Ff, parallel to the sides AB, AC, which meet the line AD, drawn through A and T.

2. From T draw the lines TB, TC, to the angular points B, C of the triangle ABC, and from e, f the parallels eG, fI; further, the lines TG, TI, then both the triangle ATG and the triangle $ATI = \frac{1}{5}$ triangle ABC.

3. In order to determine the other points of division, it is only necessary, as in the foregoing §, to cut off from the sides AB, AC, as many lines equal to AG, AI as possible, and in the case in which this can no longer be effected, or in which, as in the figure, this is impossible, the points H and K for the quadrilaterals $BGTH$, $CITK$ may be determined by § XVI, Cor. 1.

4. Since in this case the parts ATG, $BGTH$, ATI, $CITK$, collectively form $\frac{4}{5}$ of the triangle ABC, consequently the triangle HTK must be equal to the last part. If HTK be not the last part, then it is merely necessary, by § XV, to divide this triangle into as many equal parts as are necessary.

Demon. Draw the auxiliary lines AE, eB, then

$$\triangle ABE = \tfrac{1}{5} \triangle ABC \text{ (because } BE = \tfrac{1}{5} BC\text{)},$$

further, $\triangle ABE = \triangle ABe$ (because $Ee \parallel AB$)

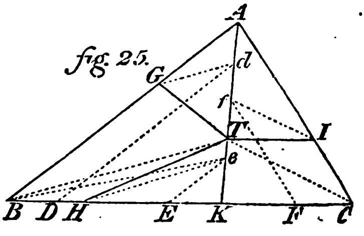
and $\triangle ABe = \triangle ATG$ (because $\triangle GBe = \triangle GTe$);

$$\text{consequently } \triangle ATG = \tfrac{1}{5} \triangle ABC.$$

The remainder of the demonstration is sufficiently clear of itself.

Cor. If a triangle is not to be divided into equal parts, but according to a given proportion, then the following method will be the best.

Thus, let the triangle ABC (*fig.* 25) be divided into four parts according to a given proportion.



1 Divide any side of the triangle, say BC, according to this proportion; let the points of division be D, E, F; through and from T draw the lines AK, TB, TC.

2. From the points of division D, E, F, draw the lines Dd, Ee, Ff, parallel to the side AB, or AC, according as they are on one or the other side of the line AK.

3. From the points d, e, f, in which these parallels meet AK, draw the lines dG, eH, fI, parallel to TB or TC; then by these means, the points G, H, I, in the sides of the triangle, will be determined.

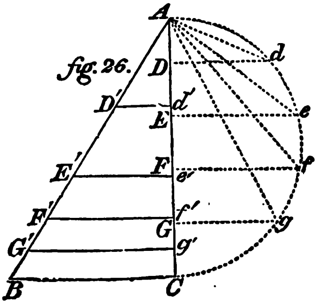
4. From T draw the lines TG, TH, TI; then the triangles ATG, ATI, and the quadrilateral figures $BGTH$, $CHTI$ are the parts of the triangle sought.

For if we suppose the lines AD, AE, AF drawn, then the triangles ABD, ADE, AEF, AFC, which collectively constitute the triangle ABC, have the required proportion, for they are to one another as their bases BD, DE, EF, FC. Now, as may in some measure be easily proved from the foregoing, the triangles ATG, ATI, are equal to the triangles ABD, AFC; and the quadrilaterals $BGTH$, $CHTI$, are equal to the triangles ADE, AEF; consequently also these parts have the required proportion.

SECTION XVIII.

Prob. *To divide a given triangle into a given number of equal parts, and in such a way, that the lines of division may be parallel to a particular side of the triangle.*

Const. Let ABC (*fig.* 26) be the given triangle; let the number of the parts into which it is required to be divided, be five, and BC the side to which the lines of division are to be parallel.



.1. Upon one of the other two sides, say AC, describe the semicircle $AdefgC$, and divide the side AC in as many equal parts, as the triangle is

to be divided in, consequently in the present case into five; the points of section are D, E, F, G.

2. From these points of division, draw the perpendicular, Dd, Ee, Ff, Gg, which meet the semicircle in the points d, e, f, g.

3. From A in AC, draw Ad, Ae, Af, Ag, then make $Ad' = Ad$, $Ae' = Ae$, and so on, and by these means determine the points d', e', f', g'.

4. From these points draw the lines $d'D'$, $e'E'$, $f'F'$, $g'G'$, parallel to the side BC, then $AD'd'$, $D'd'e'E'$, $E'e'f'F'$, $F'f'g'G'$, $G'g'CB$ are the five equal parts of the triangle ABC which were sought.

Demon. Since $AC : Ad = Ad : AD$ (*Euc.* VI. 8); then $Ad^2 = Ad'^2 = AC \times AD$ (*Euc.* VI. 17). Further, since the triangles $AD'd'$, ABC are similar, consequently

$$\triangle\, ABC : \triangle\, AD'd' = AC^2 : Ad'^2 \text{ (Euc. VI. 19)}$$
$$= AC^2 : AC\,.\,AD = AC : AD.$$

Now, since $AD = \frac{1}{5} AC$, therefore

$$\triangle\, AD'd' = \tfrac{1}{5} \triangle\, ABC.$$

In like manner it may be proved, that $\triangle AE'e' = \frac{2}{5} \triangle\, ABC$; $\triangle\, AF'f' = \frac{3}{5} \triangle\, ABC$; $\triangle\, AG'g' = \frac{4}{5} \triangle\, ABC$, from which the rest follows of course.

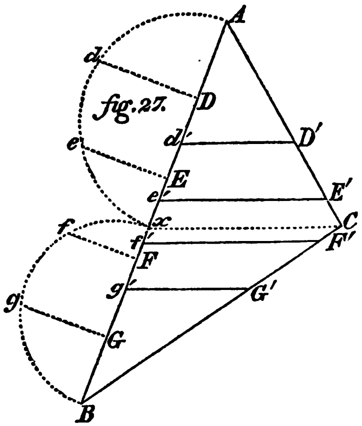
Cor. If the triangle ABC is not to be divided into equal parts, but according to a given proportion, it will merely be necessary, as may be readily seen from the above, to divide the line AC according to this proportion, and then proceed as has been already shown.

SECTION XIX.

Prob. *To divide a triangle into a given number of equal parts, in such a way, that the lines of division may be parallel to a line given in position.*

Const. Let, for instance, ABC (*fig.* 27) be the given

triangle which is to be divided into five equal parts; the lines of division are to be parallel to the line Cx.



1. Upon Ax and Bx describe the semicircles $Adex$, $Bgfx$, and divide the line AB into five equal parts, in D, E, F, G.

2. From these points of division draw the perpendiculars Dd, Ee, Ff, Gg; from A draw the lines Ad', Ae', equal to Ad, Ae, and also the lines Bg', Bf', equal to Bg, Bf.

3. From the parts d', e', f', g' draw the lines $d'D'$, E', $f'F'$, $g'G'$, parallel to Cx, then $Ad'D'$, $d'D'E'e'$, $eE'CF'f'$, $f'F'G'g'$, $g'G'B$ are the five parts sought.

Demon. By this method, the triangle ACx, by § XVIII, Cor., is divided in the same proportion as the line Ax, and also BCx in the same proportion as the line Bx. Therefore we have

1. $\triangle\, ACx : \triangle\, AD'd' = Ax : AD$.
But $\triangle\, ABC : \triangle\, ACx = AB : Ax$ (*Euc.* VI. 1);
consequently $\triangle\, ABC : AD'd' = AB : AD = 5 : 1$.
In like manner it may be proved, that
$\triangle\, ABC : \triangle\, AE'e' = AB : AE = 5 : 2$.

2. $\triangle\, BCx : \triangle\, BG'g' = Bx : BG$.
But $\triangle\, ABC : \triangle\, BCx = AB : Bx$ (*Euc.* VI. 1);
consequently $\triangle\, ABC : \triangle\, BG'g' = AB : BG : = 5 : 1$;
and thus $\triangle\, ABC : \triangle\, BF'f' = AB : BF = 5 : 2$.

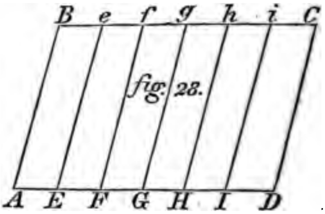
Since $\therefore$ $\triangle\, AE'e' = \frac{2}{5}\triangle\, ABC$, and $\triangle\, BF'f' = \frac{2}{5}\triangle\, ABC$, consequently the pentagon $e'E'CF'f' = \frac{1}{5}\triangle\, ABC$.

Cor. If the triangle ABC is not to be divided into equal parts, but according to a given proportion, it will only be necessary to divide the line AB according to this proportion, and then proceed as has been shown. The reason of this is readily seen.

SECTION XX.

Prob. *To divide a parallelogram into a given number of equal parts, in such a way, that the lines of division may be parallel to two opposite sides of the parallelogram.*

Const. Let $ABCD$ be the parallelogram to be divided (*fig.* 28); let the number of parts be six, and let AB, CD, be the sides to which the lines of division are parallel.



Divide one of the two other sides, say AD, into six equal parts, in E, F, G, H, I, and from these points draw the lines Ee, Ff, Gg, and so on, parallel to the sides AB, CD; then the division is done.

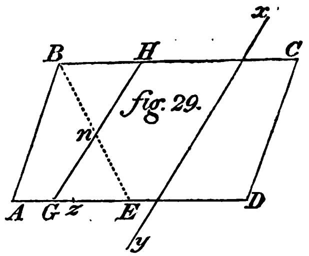
Cor. If it is required to divide the parallelogram according to a given proportion, it will merely be necessary, instead of dividing the line AB into equal parts, to divide it according to the given proportion, and then proceed as before.

SECTION XXI.

Prob. *To divide a parallelogram according to a given proportion, by a line given in position.*

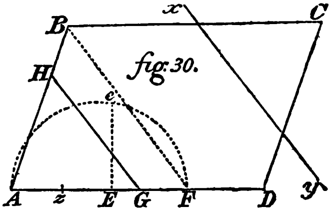
Const. Let $ABCD$ (*fig.* 29) be the parallelogram to be divided.

1. Divide one of its sides, say AD, according to the given proportion; let the point of division be z. Make $zE = Az$, and draw BE. Now if BE have the required position, then $\triangle ABE$, and quadrilateral $BCDE$ are the parts sought.



2. But if the line of division be parallel to xy, then bisect the line BE in n, and through this point draw the line GH parallel to xy, then $ABHG$ and $HCDG$ will be the required parts.

3. If the line xy have such a position, that one of the points G, H cannot fall upon the sides AD or BC, as in *fig.* 30; then draw the line BF parallel to xy, and upon AF describe the semicircle AeF; from E draw the perpendicular Ee, make $AG = Ae$, and draw GH parallel to xy; then AHG, $HBCDG$ will be the required divisions of the parallelogram.



Demon. By *Euc.* I. 41, and VI. 1, it is evident, that

$$\text{Parall. } ABCD : \triangle ABE = AD : Az,$$

whence the accuracy of 1 is evident.

Further, since (*fig.* 29) HG is bisected in n, then $\triangle BnH = \triangle EnG$; and $\therefore$ also the method in 2 is proved.

In *fig.* 30 we have

$$\text{Parall. } ABCD : \triangle ABF = 2AD : AF,$$

and by reason of the triangles ABF, AHG being similar,

$$\triangle ABF : \triangle AHG = AF^2 : AG^2 \; (= Ae^2)$$
$$= AF^2 : AF : AE,$$
$$= AF : AE;$$

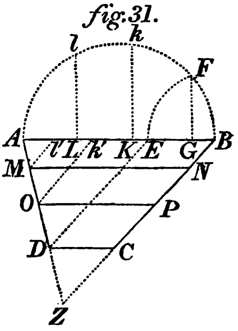
consequently, by compounding both proportions,

Parall. $ABCD : \triangle AHG = 2AD : AE = AD : Az$.

SECTION XXII.

Prob. *To divide a trapezium with two parallel sides into a given number of equal parts, so that the lines of division may be parallel to these sides.*

Const. Thus, let $ABCD$ (*fig.* 31) be the given trapezium, whose two sides AB, CD, are parallel, and which is to be divided into three equal parts.



1. Upon AB, the greater of the two parallel sides, describe the semicircle AFB; draw DE parallel to BC; and from B, with the radius BE, describe the circle EF, which cuts the semicircle in F.

2. From F draw FG perpendicular to AB, and divide the line AG into three equal parts in K and L, and from these points draw the perpendiculars Kk, Ll.

3. Upon AB, from B towards A, draw the distances Bk', Bl', equal to Bk, Bl; from these points draw the lines $k'O$, $l'M$ parallel to BC; and from the points O, M, in which these parallels meet AD, draw the lines MN, OP, parallel to AB; then $ABNM$, $MNPO$, $OPCD$, will be the three required divisions of the trapezium $ABCD$.

Demon. Produce the lines AD, BC, till they meet in Z, then the triangles DZC, OZP, MZN, AZB, are similar to one another, also.

$$DC = BE = BF,\ OP = Bk' = Bk,\ MN = Bl' = Bl.$$

We have ∴

$$\triangle\, OZP : \triangle\, DZC = OP^2 : CD^2$$
$$= Bk^2 : BF^2 = BK : BG;$$

consequently $\triangle OZP - \triangle DZC : \triangle DZC = BK - BG : BG$

or Trapez. $DOPC : \triangle DZC = GK : BG = \frac{1}{3}AG : BG$.

In like manner it may be proved, that

Trapez. $DMNC : \triangle DZC = GL : BG = \frac{2}{3}AG : BG$.

Trapez. $DABC : \triangle DZC = AG : BG = \frac{3}{3}AG : BG$.

Hence it follows, that

Trapez. $DOPC : DMNC : DABC = \frac{1}{3} : \frac{2}{3} : \frac{3}{3}$
$= 1 : 2 : 3.$

Cor. If it is required not to divide the trapezium $ABCD$ into equal parts, but according to a given proportion, it will only be necessary to divide the line AG in this proportion, and then proceed as before.

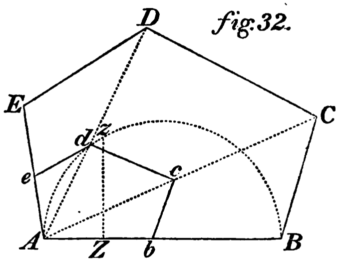
SECTION XXIII.

Prob. *To divide a given figure into two parts according to a given proportion, and in such a way, that one of the parts may be similar to the whole figure.*

Const. Let $ABCDE$ (*fig.* 32) be the given figure.

1. Divide one side of the figure, say AB, according to the given proportion; let the point of division be Z.

2. Upon AB describe the semicircle AzB, and from Z draw the perpendicular Zz, which meets the semicircle in z.



3. Make $Ab = Az$, and upon Ab describe a figure $Abcde$,

which is similar to the given one $ABCDE$; then the line $bcde$ divides the figure in the manner required.

DEMON. $ABCDE : Abcde = AB^2 : Ab^2$

$$= AB^2 : Az^2 = AB : AZ,$$

consequently $ABCDE - Abcde : Abcde = AB - AZ : AZ$, or $BCDEedcb : Abcde = ZB : AZ$.

III. QUADRATURE OF RECTILINEAR FIGURES.

SECTION XXIV.

PROB. *From the given base and altitude of a triangle to find its area.*

SOLUT. If a denote the base, h the altitude, and q the area of a triangle; then

$$q = \frac{ah}{2}.$$

This rule is well known, and is given in all elementary books on Geometry.

COR. From this equation we obtain

$$h = \frac{2q}{a},\ a = \frac{2q}{h}.$$

By means of the first of these two formulæ, the altitude of a triangle may be determined, when the area and the base are given, and by means of the second, the base of a triangle may be found, when its area and altitude are given.

EXAM. 1. The base of a triangle measures $325^0.\ 7'.\ 9''$, and its altitude $67^0.\ 8'.\ 3''$: what is its area? Ans. $11049.16785\ \square^0$, or $11049\ \square^0\ 16\ \square'\ 78\ \square''\ 50\ \square'''$.

EXAM. 2. The base of a triangle measures $763^0.\ 0'.\ 5''$, and its altitude $9'.\ 3''.\ 7'''$: what is its area? Ans. $357.488925\ \square^0$, or $357\ \square^0\ 48\ \square'\ 89\ \square''\ 25\ \square'''$.

Exam. 3. The area of a triangle is 7325 $\square^0$ 26 $\square'$, and its base $58^0.\ 9'.\ 7''$: what is its altitude? Ans. $248^0{\cdot}4402...$, or $248^0.\ 4'.\ 4''.\ 0'''.\ 2^{\text{IV}}...$

Exam. 4. The area of a triangle is 62583 $\square^0$ 04 $\square'$ 79 $\square''$, and its altitude $127^0.\ 5'$: what is its base? Ans. $981^0{\cdot}6948...$, or $981^0.\ 6'.\ 9''.\ 4'''.\ 8^{\text{IV}}$.

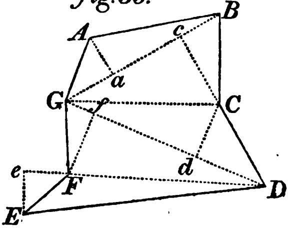
SECTION XXV.

Prob. *To calculate the area of any polygon.*

Solut. Divide the whole of the polygon into triangles by diagonals, calculate all these triangles, by multiplying the base of each by half its altitude, and by adding all the results thus obtained together; we then get the area of the polygon.

Cor. In order to shorten the operation, a common base can always be given, where it is practicable, to two triangles. Thus the polygon $ABCDEFG$ (*fig.* 33) may be divided into the triangles ABG, BCG, DCG, DFG, DEF, of which the first and second have BG, and third and fourth DG, for a common base. When the necessary perpendiculars are drawn, we then have,

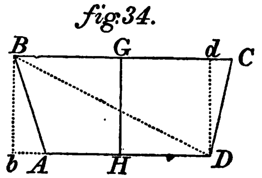
fig. 33.



$$\text{Pol. } ABCDEFG = \tfrac{1}{2}\,[Aa\,.\,BG + Cc\,.\,BG + Cd\,.\,DG + Ff\,.\,DG + Ee\,.\,DF.]$$
$$= \tfrac{1}{2}\,[(Aa+Cc)BG + (Cd+Ff)DG + Ee\,.\,DF].$$

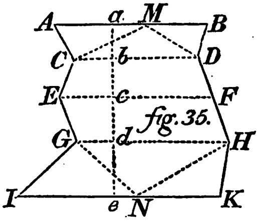
Exam. Let $BG = 61^0.\ 5'$, $DG = 75^0.\ 9'.\ 3''$, $DF = 67^0.\ 3'$, $Aa = 15^0.\ 7'$, $Cc = 28^0.\ 0'.\ 9''$, $Cd = 21^0.\ 1'.\ 7''$, $Ff = 22^0$, $Ee = 16^0.\ 8'$: what is the area of the heptagon $ABCDEFG$? Ans. $3550{\cdot}81155$ $\square^0$, or 3550 $\square^0$ 81 $\square'$ 15 $\square''$ 50 $\square'''$.

Remark. A trapezium $ABCD$ (*fig.* 34) with two parallel sides AD, BC, may always be divided by the diagonal BD, into two triangles ABD, BDC, whose bases are these sides of the trapezium, and whose altitudes Bb, Dd, are equal to its altitude GH. Therefore in this case,



$$\text{Trapez. } ABCD = \tfrac{1}{2}\, GH\,(AD + BC).$$

If the whole of a polygon can be divided into such trapeziums, then its area may more easily be calculated. This is the case in *fig.* 35, when the lines AB, CD, EF, GH, IK, are all parallel to one another. Thus then, when ae is perpendicular to these sides, the area of the polygon =



$$\tfrac{1}{2}\begin{bmatrix} ab\,(AB + CD) + bc\,(CD + EF) \\ + cd\,(EF + GH) + de\,(GH + IK) \end{bmatrix}.$$

If the points A, B meet in M, as also the points I, K in N, so that, instead of the trapeziums $ABDC$, $GHKI$, we have the triangles CMD, GNH: then both AB and $IK=0$, and we find the area of the figure =

$$\tfrac{1}{2}\begin{bmatrix} ab\,.\,CD + bc\,(CD + EF) \\ + cd\,(EF + GH) + de\,.\,GH \end{bmatrix}.$$

If, besides, the distances between all these parallels are the same, or $ab = bc = cd = de$, then the area =

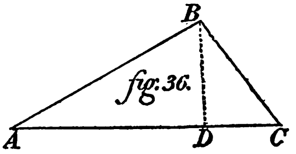
$$ab\,(CD + EF + GH),$$

Consequently we find the area of the figure, by multiplying the sum of all the parallels by the equal distances.

SECTION XXVI.

Prob. *From the two sides of a triangle, and the angle contained by them, to find the area of the triangle.*

Const. Let ABC be the given triangle (*fig.* 36), of which the sides AB, AC, and the angle BAC, are given: find its area. Let $AC=a$, $AB=b$, $\angle\, BAC = \alpha$, and the required area of the triangle $= q$.



1. Draw the perpendicular BD: then

$$BD = b\ Sin.\ \alpha.$$

2. Therefore $\triangle ABC = \frac{AC \cdot BD}{2} = \frac{ab \, Sin. \, \alpha}{2}$

or $$q = \frac{ab \, Sin. \, \alpha}{2}$$

COR. 1. Hence it follows: that two triangles, which contain one angle in the one equal to one angle in the other, are as the products of the sides which include these triangles.

Since every parallelogram is divided into two equal parts by its diagonal; consequently this rule likewise obtains for parallelograms which have one equal angle.

COR. 2. From the equation $q = \frac{ab \, Sin. \, \alpha}{2}$, we get

$$a = \frac{2q}{b \, Sin. \, \alpha}, \; b = \frac{2q}{a \, Sin. \, \alpha}, \; Sin. \, \alpha = \frac{2q}{ab}.$$

The calculation of all these formulæ is most easily effected by means of logarithms.

EXAM. 1. One side of a triangle measures $257'. \, 9''$, the other $356'. \, 3''$, and the angle contained by these sides $25^0. \, 13'$: what is the area of this triangle? Ans. $19574{\cdot}46 \, \square'$, or $19574 \, \square' \, 46 \, \square''$.

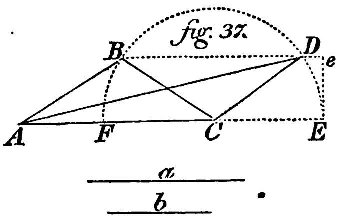
EXAM. 2. The area of a triangle is $27534 \, \square'$, one of its sides $67'. \, 3''$, and one of the two angles, which are adjacent to this side, $121^0. \, 5'$: what is the other side which is adjacent to this angle? Ans. $955'. \, 4''. \, 3'''$.

EXAM. 3. The area of a triangle is $1254 \, \square' \, 26 \, \square''$, one of its sides $= 138'$, and the other $= 59'$: what is the angle included by these two sides? Ans. $17^0. \, 56'. \, 40''. \, 3$, or $162^0. \, 3'. \, 19'', \, 7$.

REMARK. The double values of the angle in the last example arises from this; because two angles which together are equal to 180°, belong to the same sine. Consequently there are always two triangles, having a given area and two given sides, which may be easily represented algebraically.

Let a and b (*fig.* 37) be the given sides, and the given area of the triangle

$=q$. Assume one of these two lines, say a, as the base of the triangle, and make $AC=a$; with the radius $CF=b$, describe the semicircle $FBDE$, and from E, in which it meets AC produced, draw the perpendicular $Ee=\frac{2q}{a}$, then Ee is the altitude of the triangle (§ XXIV, Cor.). Then draw eB parallel to AE, and from the points B, D, in which these cut the semicircle, the lines CB, CD, AB, AD; then ABC, ADC are the required triangles, because they have the required altitude, and the given sides. Now, in the isosceles triangles BCD, the angles CBD, CDB, are equal; further, because eB is parallel to AE, $DBC = BCA$, and $BDC = DCE$; consequently also $BCA = DCE$, and $\therefore ACD + ACB = ACD + DCE = 2R$.



SECTION XXVII.

Prob. *From the two given angles of a triangle, and one of its sides, to find its area.*

Solut. In the triangle ABC (*fig.* 36) there are two angles given, consequently also the third, together with the side AC; required to find its area. Let $BAC=\alpha$, $ABC=\beta$, $ACB=\gamma$, $AC=a$, and the required area $=q$. Draw BD perpendicular to AC: then

1. $Sin.\ \beta : Sin.\ \gamma = a : AB$

$$\therefore AB = \frac{a\ Sin.\ \gamma}{Sin.\ \beta}$$

2. In the right-angled triangle ABD, whose right angle is at D,

$$BD = AB\ Sin.\ \alpha = \frac{a\ Sin.\ \alpha\ Sin.\ \gamma}{Sin.\ \beta}$$

3. Therefore $q = \frac{1}{2} AC . BD = \dfrac{a^2\ Sin.\ \alpha\ Sin.\ \gamma}{2\ Sin.\ \beta}$

This formula is most easily calculated by means of logarithms.

Cor. If the triangle be an isosceles one, then we have

1. When $\beta = \gamma$, $q = \frac{1}{2} a^2$ *Sin.* α.

2. When $\alpha = \gamma$, then $\beta = 180^0 - 2\alpha$, and therefore *Sin.* β = *Sin.* 2α = 2 *Sin.* α *Cos.* α, consequently

$$q = \frac{a^2 \, Sin. \, \alpha}{2 \, Cos. \, \alpha} = \tfrac{1}{2} a^2 \, Tang. \, \alpha.$$

Exam. 1. When $\alpha = 38^0. 40'$, $\beta = 83^0. 30'$, consequently $\gamma = 57^0. 50'$, and $a = 120'$; then the area of the triangle is 3832.61 $\square'$.

Exam. 2. When $\alpha = 128^0. 25'$, $\beta = 27^0. 5'$, $\therefore \gamma = 24^0. 30'$, and $a = 135'$: then the area of the triangle is 6503·19 $\square'$.

Exam. 3. When $\alpha = 37^0. 5'. 24''$, $\beta = 67^0. 45'. 23''$, $\therefore \gamma = 75^0. 9'. 13''$, and $a = 435''$: then the area of the triangle is 59587·36 $\square'$.

SECTION XXVIII.

Prob. *To find the area of a triangle from its angles and altitude.*

Solut. In the triangle ABC (*fig.* 36) let the angle $BAC = \alpha$, $ABC = \beta$, $ACB = \gamma$, and the altitude $BD = h$ be given; let the required area $= q$: then

1. By § 27, $$q = \frac{AC^2 \, Sin. \, \alpha \, Sin. \, \gamma}{2 \, Sin. \, \beta}.$$

2. By § 24, Cor., $AC = \dfrac{2q}{h}$.

3. If we substitute 2 in 1, we then get

$$q = \frac{2q^2 \, Sin. \, \alpha \, Sin. \, \gamma}{h^2 \, Sin. \, \beta},$$

and $\therefore$, $$q = \frac{h^2 \, Sin. \, \beta}{2 \, Sin. \, \alpha \, Sin. \, \beta}.$$

Exam. When $h = 357\frac{1}{2}'$, $\alpha = 31^{\circ}.\ 13'.\ 7''$, $\beta = 106^{\circ}.\ 41'.\ 53''$, then $q = 176203{\cdot}6 \,\square'$.

SECTION XXIX.

Prob. *To find the area of a triangle from its three sides.*

Solut. In the triangle ABC (*fig.* 36), the three sides AB, AC, BC are given, and its area is sought. Let $AB = a$, $AC = b$, $BC = c$, and the area $= q$.

1. We know from Trigonometry, that

$$c^2 = a^2 + b^2 - 2ab \text{ Cos. } A$$

$$\therefore \qquad \text{Cos. } A = \frac{a^2 + b^2 - c^2}{2ab}.$$

2. Hence we obtain,

$$1 + \text{Cos. } A = 1 + \frac{a^2 + b^2 - c^2}{2ab} = \frac{(a + b)^2 - c^2}{2ab}$$

$$= \frac{(a + b + c)(a + b - c)}{2ab}$$

$$1 - \text{Cos. } A = 1 - \frac{a^2 + b^2 - c^2}{2ab} = \frac{c^2 - (a - b)^2}{2ab}$$

$$= \frac{(c + a - b)(c - a + b)}{2ab}$$

3. The multiplication of these equations gives

$$1 - \text{Cos.}^2 A = \frac{(a + b + c)(a + b - c)(c + a - b)(c - a + b)}{4a^2b^2}$$

or, when we substitute $\text{Sin.}^2 A$ for $1 - \text{Cos.}^2 A$, and K for the number of the fraction on the right side of the equation,

$$\text{Sin.}^2 A = \frac{K}{4a^2b^2}, \text{ and } \text{Sin. } A = \frac{\sqrt{K}}{2ab}.$$

4. But by § 26, $q = \frac{ab \, Sin. \, A}{2}$; we have therefore

$$q = \frac{\sqrt{K}}{4} = \tfrac{1}{4}\sqrt{(a+b+c)\ (a+b-c)\ (c+a-b)\ (c+b-a)}.$$

From this formula we deduce the following rules for finding the area of a triangle from its sides :

1. *Add all the sides together.*

2. *From the sum of every two sides subtract the third.*

3. *Multiply the sum obtained from* 1 *by the three remainders obtained from* 2.

4. *From the product subtract the square root, and divide this root by* 4.

The actual calculation, when a, b, c, are not very small numbers, is most easily performed by means of logarithms.

Cor. When the triangle is an equilateral one, $a = b = c$; consequently

$$q = \frac{\sqrt{3a^4}}{4} = \frac{a^2\sqrt{3}}{4}$$

When it is an isosceles triangle, let $c = b$, then

$$q = \frac{\sqrt{a^2\,(2b+a)\ (2b-a)}}{4} = \frac{a}{4}\sqrt{(2b+a)\ (2b-a)}$$

Exam. When $a = 563'$, $b = 295'$, $c = 387'$, then $q = 53447{\cdot}73 \,\square'$.

SECTION XXX.

Prob. *From the four given sides of a trapezium, of which two are parallel to one another, to find its area.*

Solut. Let *AB*, *DC* (*fig.* 38) be the two parallel sides

of the trapezium $ABCD$, and $AB = a$, $BC = b$, $CD = c$, $DA = d$. Draw BE parallel to AD, and for shortness sake, put $CE = c - a = f$: then

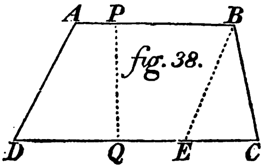


fig. 38.

1. In the triangle BEC, $BC = b$, $CE = f$, $BE = AD = d$; consequently, by the foregoing §,
$\Delta BEC = \frac{1}{4}\sqrt{(b+d+f)\ (b+d-f)\ (b+f-d(\ (d+f-b)}$.

2. Draw the perpendicular PQ: then this is the altitude both of the trapezium and the triangle; we have ∴

Trapez. $ABCD = \frac{1}{2} PQ\ (AB + CD)$ (§ 25, Remark.)
and $\quad \Delta BEC = \frac{1}{2} PQ \,.\, CE$, consequently
Trapez. $ABCD : \Delta BEC = \frac{1}{2} PQ\ (AB + CD) : \frac{1}{2} PQ \,.\, CE$
$= AB + CD : CE = a + c : f$
and ∴

$$\text{Trapez. } ABCD = \frac{a+c}{f} \Delta\, BEC =$$
$$\frac{a+c}{4f} \sqrt{(b+d+f)\ (b+d-f)\ (b+f-d)\ (d+f-b)}$$

Exam. When $a = 324'$, $b = 137'$, $c = 431'$, $d = 122'$, then $q = 44079{\cdot}76\ \square'$.

SECTION XXXI.

Prob. *All the sides of a quadrilateral, whose opposite angles are together equal to two right angles (about which, consequently, a circle may be described), are given: required to find its area.*

Solut. Let $ABCD$ (*fig.* 39) be the given quadrilateral, in which, agreeably to the hypothesis, $ABC + ADC = 2R$. Let $AB = a$, $BC = b$, $CD = c$, $DA = d$. The unknown angle $ADC = \phi$, and ∴ $ABC = 180^0 - \phi$. Draw the diagonal AC; then

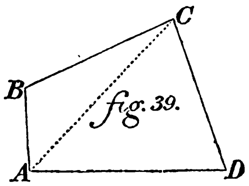


fig. 39.

1. In the triangle ADC,

$$AC^2 = c^2 + d^2 - 2\,cd\ Cos.\ \phi$$

and in triangle ABC

$$AC^2 = a^2 + b^2 + 2\,ab\ Cos.\ \phi$$

(because $Cos.\ ABC = Cos.\ (180^0 - \phi) = -\ Cos.\ \phi.$) We have .·.

$$a^2 + b^2 + 2\,ab\ Cos.\ \phi = c^2 + d^2 - 2\,cd\ Cos.\ \phi$$

and consequently $Cos.\ \phi = \dfrac{c^2 + d^2 - a^2 - b^2}{2\,ab + 2\,cd}$

2. Hence we get

$$1 + Cos.\ \phi = \frac{c^2 + 2\,cd + d^2 - a^2 + 2\,ab - b^2}{2\,ab + 2\,cd}$$

$$= \frac{(c + d)^2 - (a - b)^2}{2\,ab + 2\,cd}$$

$$= \frac{(c + d + a - b)\ (c + d - a + b)}{2\,ab + 2\,cd}$$

further $1 - Cos.\ \phi = \dfrac{a^2 + 2\,ab + b^2 - c^2 + 2\,cd - d^2}{2\,ab + 2\,cd}$

$$= \frac{(a + b)^2 - (c - d)^2}{2\,ab + 2\,cd}$$

$$= \frac{(a + b + c - d)\ (a + b - c + d)}{2\,ab + 2\,cd}$$

3. The multiplication of these two equations gives

$$1 - Cos.^2\ \phi = Sin.^2\ \phi =$$

$$\frac{(a + b + c - d)\ (a + b - c + d)\ (c + d - a + b)\ (c + d + a - b)}{(2\,ab + 2\,cd)^2}$$

or, when we substitute K for the numerator of the fraction and extract the root from both sides,

$$Sin.\ \phi = \frac{\sqrt{K}}{2\,ab + 2\,cd}$$

4. Now $\triangle\ ADC = \dfrac{cd\ Sin.\ \phi}{2}$

and $\triangle ABC = \frac{ab\ Sin.\ (180^0 - \phi)}{2} = \frac{ab\ Sin.\ \phi}{2};$

consequently trapez. $ABCD = \frac{(ab + cd)\ Sin.\ \phi}{2},$

or, when for *Sin.* ϕ we substitute its value in 3,

$$\text{Trapez. } ABCD = \frac{\sqrt{K}}{4} =$$

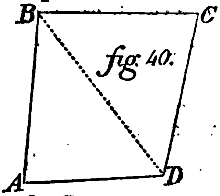
$$\tfrac{1}{4}\sqrt{(a+b+c-d)\,(a+b+d-c)\,(c+d+a-b)\,(c+d+b-a)}$$

The rule obtained from this formula for the calculation of this kind of quadrilaterals, may be expressed in words in the following way:

Add every three sides of the figure together, and always subtract the fourth from their sum; then multiply the four remainders together, extract the root from the product, and divide the root by 4.

SECTION XXXII.

Prob. *To find the area of a quadrilateral, in which two opposite angles are equal, from its four sides.*



Const. Let $ABCD$ (*fig.* 40) be a quadrilateral, in which the angles BAD, BCD are equal, and $AB=a$, $BC=b$, $CD=c$, $DA=d$, the required area $=q$. Put the unknown angle $BAD=BCD=\phi$: then in the $\triangle$ BAD

$$BD^2 = a^2 + d^2 - 2ad\ Cos.\ \phi$$

and in the $\triangle BCD$, $BD^2 = b^2 + c^2 - 2bc\ Cos.\ \phi$

consequently $a^2 + d^2 - 2ad\ Cos.\ \phi = b^2 + c^2 - 2bc\ Cos.\ \phi$

and $\therefore$ $Cos.\ \phi = \frac{a^2 + d^2 - b^2 - c^2}{2ad - 2bc}.$

If we proceed now as in the foregoing §, we at last obtain

$$q = \tfrac{1}{4}\frac{ad + bc}{ad - bc}\sqrt{\begin{bmatrix}(a + b + c + d)\,(a + b - c - d)\\(a + d - b - c)\,(b + d - a - c)\end{bmatrix}}.$$

Exam. When $a=37^0$, $b=31^0.\ 7'$, $c=29^0$, $d=16^0.\ 14'$: then $q = 713{\cdot}0677\ \square^0$.

Remark. In this example, two of the factors, which are under the radical sign of the expression found for q, are negative; for we have $a+d-b-c=-8.\ 3$, and $b+d-a-c=-16.\ 9$. But since two negative magnitudes give a positive product, we consequently can omit entirely the sign $-$. Further, for this example, the denominator of the fraction $\frac{ad+bc}{ad-bc}$, is negative, and consequently the fraction itself; but in this case also it is not necessary to retain the negative sign, because the magnitude under the radical sign may be assumed to be either positive or negative. The problem may also be impossible, as, for instance, when we assume $a=17^\circ$, $b=23^\circ$, $c=27^\circ$, $d=40^\circ$.

SECTION XXXIII.

Prob. *To find the area of a triangle from its sides, when one of its angles is a right angle.*

Const. Let $ABCD$ (*fig.* 41) be the quadrilateral, and BAD a right angle. Further, let $AB=a$, $BC=b$, $CD=c$, $DA=d$, and the required area $=q$: then

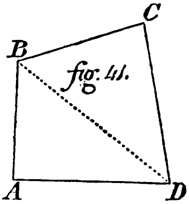


fig. 41.

1. $BD^2=a^2+d^2$ (*Euc.* I. 47), or $f^2=a^2+d^2$, when, for the sake of brevity, we put $BD=f$.

2. In the $\triangle\ BCD$ ∴ all the three sides are known; we consequently have, by § XXIX,

$$\triangle BCD=\tfrac{1}{4}\sqrt{(b+c+f)\ (b+c-f)\ (b+f-c)\ (c+f-b)}$$

3. Since BA is perpendicular to AD, the

$$\triangle\ BAD=\tfrac{1}{2}\,ad,$$

consequently, because trapez. $ABCD=\triangle BAD+\triangle BCD$

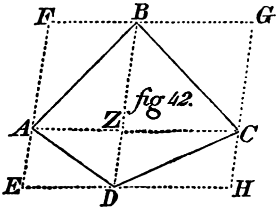
$$q=\tfrac{1}{2}ad+\tfrac{1}{4}\sqrt{(b+c+f)\ (b+c-f)\ (b+f-c)\ (c+f-b)}$$

Exam. When $a=28'$, $b=32'$, $c=41'$, $d=39'$, then $f=48{\cdot}0104$, and $q=1194{\cdot}3332\ \square'$.

SECTION XXXIV.

Prob. *To find the area of a quadrilateral from its two diagonals, and the angle which they include.*

Solut. Let $ABCD$ (*fig.* 42) be the given quadrilateral and $AC = a$, $BD = b$, $\angle AZD = \alpha$.



1. Through A, B, draw the lines EF, HG, parallel to the diagonal BD, and through B, D the lines FG, EH, parallel to AC; then $AFGC$, $AEHC$ are two parallelograms, and the triangles ABC, ADC are their halves. The quadrilateral $ABCD$ is $\therefore$ the half of the parallelogram $EFGH$, and consequently, if we suppose the line FH drawn, equal to the triangle FEH.

2. But by § XXVI, since $EH = AC = a$, $FE = BD = b$, and $\angle FEH = AZD = \alpha$,

$$\triangle FEH = \frac{ab \text{ Sin. } \alpha}{2};$$

consequently also Trapez. $ABCD = \dfrac{ab \text{ Sin. } \alpha}{2}$.

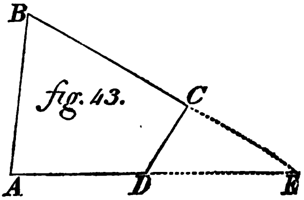
Cor. Consequently every quadrilateral is equal to a triangle, in which two sides are equal to the two diagonals of the quadrilateral and the angle included by them is equal to the angle of the quadrilateral opposite their intersection. In this case it is also immaterial, which of the two angles AZD, DZC is taken for the angle of the triangle, because both give equal triangles (§ XXVI, Remark).

SECTION XXXV.

Prob. *In a quadrilateral, three angles, consequently also the fourth, and two opposite sides, are given: required to find its area.*

Solut. Let $ABCD$ (*fig.* 43) be the quadrilateral figure,

and AB, DC, the given sides. Let $AB = a$, $CD = b$, $BAD = \alpha$, $ABC = \beta$, $BCD = \gamma$, and $ADC = 360^0 - (\alpha + \beta + \gamma) = \delta$; the required area $= q$. Produce the two unknown sides AD, BC, till they meet in E.



1. In the triangle ABE, the two angles $EAB = \alpha$, $EBA = \beta$, and the side $AB = a$ are given; we have $\therefore$ by § XXVII,

$$\triangle AEB = \frac{a^2 \, Sin. \, \alpha \, Sin. \, \beta}{2 \, Sin. \, (\alpha + \beta)},$$

(because $Sin. \, AEB = Sin. \, [180^0 - (\alpha + \beta)] = Sin. \, (\alpha + \beta)$.

2. In like manner, we find

$$\triangle CED = \frac{b^2 \, Sin. \, \gamma \, Sin. \, \delta}{2 \, Sin. \, (\alpha + \beta)},$$

(because $CD = b$, $Sin. \, DCE = Sin. \, (180^0 - \gamma) = Sin. \, \gamma$, $Sin. \, CDE = Sin. \, (180^0 - \delta) = Sin. \, \delta$).

3. Now since Trapez. $ABCD = \triangle AEB - \triangle CED$; therefore

$$q = \frac{a^2 \, Sin. \, \alpha \, Sin. \, \beta - b^2 \, Sin. \, \gamma \, Sin. \, \delta}{2 \, Sin. \, (\alpha + \beta)}$$

REMARK. The calculation will be most easily performed by finding each of the triangles AEB, CED separately, and subtracting the areas thus found from one another.

EXAM. Let $a = 536'$, $b = 379'$, $\alpha = 83^0. \, 28'$, $\beta = 69^0. \, 34'$, $\gamma = 102^0. \, 20'$: then $q = 145209 \cdot 1 \, \square'$:

SECTION XXXVI.

PROB. *To find the area of a trapezium with two parallel sides, when its altitude, one of the parallel sides, and the two angles adjacent to it are given.*

= 28'

SOLUT. L... ...CD (*figs.* 44, 45) be a trapezium, having

two parallel sides AD, BC, and PQ perpendicular to these sides. Let the side $AD=a$, the altitude $PQ = h$, and the $\angle\ BAD = \alpha$, $ADC = \delta$, the required area $= q$.

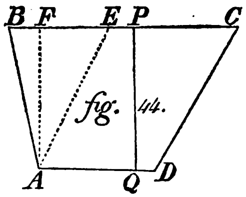


fig. 44.

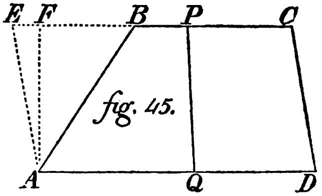


fig. 45.

Draw AE parallel to DC, which intersects the line BC in E. This point falls either on the line BC itself (*fig.* 44), or upon BC produced (*fig.* 45). The first case obtains, when $\alpha+\delta > 2R$, the second, when $\alpha + \delta < 2R$.

First Case. 1. In the $\triangle\ ABE$ (*fig.* 44), the $\angle\ ABE = 180^0 - \alpha$, $AEB = BCD = 180^0 - \delta$, $BAE = BAD + ADC - (EAD + ADC) = \alpha + \delta - 180^0$; consequently $Sin.\ ABE = Sin.\ \alpha$, $Sin.\ AEB = Sin.\ \delta$, $Sin.\ BAE = - Sin.\ (\alpha + \delta)$, and $\therefore$ (§ XXVIII)

$$\triangle\ ABE = -\frac{h^2 Sin.\ (\alpha + \delta)}{2\ Sin.\ \alpha\ Sin.\ \delta}.$$

2. The area of the parallelogram $AECD = ah$. Now, since Trapez. $ABCD = AECD + \triangle\ ABE$; then

$$q = ah - \frac{h^2\ Sin.\ (\alpha + \delta)}{2\ Sin.\ \alpha\ Sin.\ \delta}.$$

Second Case. 1. In the $\triangle\ ABE$ (*fig.* 45), $ABE = BAD = \alpha$, $AEB = ADC = \delta$, $BAE = 180^0-(\alpha+\delta)$; consequently $Sin.\ ABE = Sin.\ \alpha$, $Sin.\ AEB = Sin.\ \delta$, $Sin.\ BAE = Sin.\ (\alpha+\delta)$ and therefore (§ XXVIII.)

$$\triangle\ ABE = \frac{h^2\ Sin.\ (\alpha + \delta)}{2\ Sin.\ \alpha\ Sin.\ \delta}.$$

2. The area of the parallelogram $AECD = ah$. Now since Trapez. $ABCD = AECD - \triangle\ ABE$; therefore

$$q = ah - \frac{h^2\ Sin.\ (\alpha + \delta)}{2\ Sin.\ \alpha\ Sin.\ \delta}.$$

We find $\therefore$ for the area of the trapezium, one and the

same expression, whether the point E be in the line BC, or this line produced, as might, indeed, have been expected from the generality of the trigonometrical and algebraical formulæ.

Cor. When $\alpha + \delta = 180^0$, then $Sin. (\alpha + \delta) = 0$, and $q = ah$, which indeed must be the case, because in this case the trapezium is transformed into a parallelogram.

When $\alpha + \delta = 90^0$, $Sin. (\alpha + \delta) = 1$, and $Sin. \delta = Sin. (90^0 - \alpha) = Cos. \alpha$; consequently

$$q = ah - \frac{h^2}{2\, Sin.\, \alpha\, Cos.\, \alpha} = ah - \frac{h^2}{Sin.\, 2\, \alpha}$$

Exam. 1. Let $\alpha = 117^0.\ 36'$, $\delta = 135^0.\ 29'$, $a = 257'$, $h = 87'$. Here $\alpha + \delta = 253^0.\ 5'$, and $Sin. (a + \delta) = - Sin. 73^0.\ 5'$; further $Sin. \alpha = Sin. 62^0. 24'$, $Sin. \delta = Sin. 44^0. 31$; $\therefore$

$$q = 257 \cdot 87 + \frac{87^2 . Sin. 73^0.\ 5'}{2\, Sin. 62^0.\ 24'. Sin. 44^0. 31'} = 28186 \cdot 38\ \square'.$$

Exam. 2. When $\alpha = 37^0.\ 18'$, $\delta = 52^0.\ 42'$, $a = 350'$, $h = 34'$, then $q = 10700 \cdot 95\ \square'$.

SECTION XXXVII.

Prob. *In a quadrilateral three of the sides in succession, and the angles included by them, are given: required to find its area.*

Solut. In the quadrilateral $ABCD$ (*fig.* 43) three sides are given, viz. $CB = a$, $BA = b$, $AD = c$, and the angles $DAB = \alpha$, $ABC = \beta$. Produce the sides AD, BC, till they meet in E.

1. In the triangle ABE, the angles DAB, ABC, and the side AB, are given; we have $\therefore$

$$BE = \frac{b\, Sin.\, \alpha}{Sin. (\alpha + \beta)},\ AE = \frac{b\, Sin.\, \beta}{Sin. (\alpha + \beta)}$$

$$\triangle ABE = \frac{b^2\, Sin.\, \alpha\, Sin.\, \beta}{2\, Sin. (\alpha + \beta)}\ (\S\ XXVII).$$

2. Hence we obtain

$$CE = BE - BC = \frac{b \; Sin. \; \alpha}{Sin. \; (\alpha + \beta)} - a$$

$$DE = AE - AD = \frac{b \; Sin. \; \beta}{Sin. \; (\alpha + \beta)} - c$$

and ∴

$$\triangle CDE = \tfrac{1}{2}\left[\frac{b \; Sin. \; \alpha}{Sin. \; (\alpha+\beta)} - a\right]\left[\frac{b \; Sin. \; \beta}{Sin. \; (\alpha+\beta)} - c\right] Sin. (\alpha+\beta)$$

(§ XXVI).

3. Consequently

Trapez. $ABCD = \triangle \; ABE - \triangle CDE =$
$\tfrac{1}{2}\left[ab \; Sin. \; \beta + bc \; Sin. \; \alpha - ac \; Sin. \; (\alpha + \beta)\right]$.

Cor. If AD be parallel to BC, then we have $\alpha+\beta=180^{0}$, and $Sin. \; \beta = Sin. \; \alpha$, $Sin. \; (\alpha + \beta) = 0$; ∴

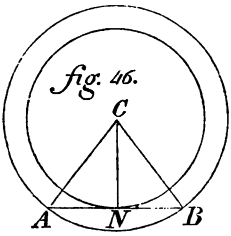
Trapez. $ABCD = \tfrac{1}{2} b \; (a + c) \; Sin. \; \alpha$.

Exam. Let $a=287'. \; 3$, $b=205'$, $c=167'. \; 4$, $\alpha=75^{0}. \; 13'$, $\beta = 49^{0}. \; 36'$. Here we find $\tfrac{1}{2} \; ab \; Sin. \; \beta = 22425{\cdot}96$; $\tfrac{1}{2} \; bc \; Sin. \; \alpha = 16590{\cdot}51$; $\tfrac{1}{2} \; ac \; Sin. \; (\alpha + \beta) = 19742{\cdot}19$; ∴ Trapez. $ABCD = 19274{\cdot}28 \; \square'$.

SECTION XXXVIII.

Prob. *To find the area of a regular polygon, from the number of its sides, and the radius of the circle described about it, or within it.*

Solut. Let AB (*fig.* 46) be the side of a regular polygon of n sides, the radius of a circle described about it, $CA = CB = r$, the radius of the circle described within it $CN = \rho$, and the area of the polygon $= P$.



1. Since the polygon has n sides the $\angle \; ACB = \frac{360^{0}}{n}$, and ∴ (§ XXVI),

$$\triangle\ ACB = \tfrac{1}{2}\, r^2\ Sin.\ \frac{360^0}{n}$$

consequently $P = \frac{1}{2}\, nr^2\ Sin. \frac{360^0}{n}$.

2. Since $ACB = \frac{360^0}{n}$, therefore $ACN = \frac{1}{2} ACB = \frac{180^0}{n}$,

$AN = \rho\,\text{Tan.}\,\frac{180^0}{n}$, $AB = 2\ AN = 2\rho\ \text{Tan.}\,\frac{180^0}{n}$; $\therefore$

$$\triangle\ ACB = \rho^2\ \text{Tan.}\ \frac{180^0}{n}$$

$$P = n\rho^2\,\text{Tan.}\,\frac{180^0}{n}.$$

Exam. 1. What is the area of a Nonagon, when the radius of the circle described about it, is $5''.\ 8'''$? Ans. $97{\cdot}3052\ \square''$.

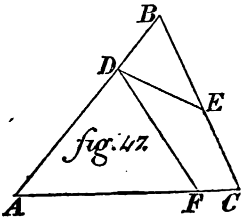
Exam. 2. What is the area of a Quindecagon, when the radius of the circle described within it is $9''$? Ans. $252{\cdot}6127\ \square''$, or $2{\cdot}526127\ \square'$.

IV. PARTITION OF FIGURES BY ALGEBRA.

SECTION XXXIX.

Prob. *From a given point in one of the sides of a triangle, to divide it in a given proportion.*

Solut. Suppose the triangle ABC (*fig.* 47), from the point D, divided by a line DE in such a way, that the whole is to the part DBE, as $m : n$. Let $AB=a$, $BC=b$, $AC=c$, $BD=d$.



1. If we knew how to determine the point E, we could draw the line DE. Let $\therefore BE=x$. By § XXVI, Cor. 1,

$$\triangle ABC : \triangle BDE = AB \,.\, BC : BD \,.\, BE,$$

or

$$m : n = ab : dx,$$

$$\therefore \quad x = \frac{nab}{md}$$

2. If in the calculation of this expression, BE is found to be greater than BC, this indicates that the line of division does not meet the side BC. Let $\therefore DE$ be the line of division, and $AF=y$; then again, by § XXVI. Cor. 1.

$$\triangle ABC : \triangle ADF = AB \,.\, AC : AD \,.\, AF$$
$$= ac : (a-d)\,y$$

But according to the hypothesis,

$$\triangle ABC : BDFC = m : n$$

and $\therefore$ $\quad \triangle ABC : \triangle ADF = m : m-n$

consequently $\quad m : m-n = ac : (a-d)\,y$

and $\quad y = \dfrac{(m-n)\,ac}{m\,(a-d)}$

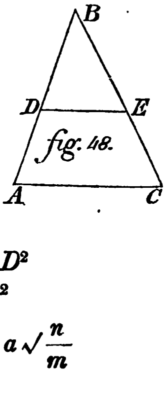
EXAM. 1. Let $AB = 74'$, $BC = 47'$, $AC = 68'$, $BD = 19'.\ 3''$: from the point D in the triangle ABC, from B towards C, cut off a part, which is to the whole triangle, as $7'$: 48. In this case the point E is in BC, and $BE = 26'{\cdot}2802$.

EXAM. 2. But if from this triangle the third part be cut off, then the point of section is in the line AC, and then we must assume $AF = 61'{\cdot}3284$.

SECTION XL.

PROB. *To divide a triangle in a given proportion by a line which is parallel to one of its sides.*

SOLUT. Let the triangle ABC (*fig.* 48) be divided by a line DE, which is parallel to AC, in such a way, that the whole triangle is to the section DBE, as $m : n$.



Since this is merely to determine the point D or E, from which the line DE is to be drawn; let $AB = a$, $BD = x$.

1. Because the triangles ABC, DBE, are similar,

$$\triangle ABC : \triangle DBE = AB^2 : BD^2$$

or

$$m : n = a^2 : x^2$$

consequently

$$x = \sqrt{\frac{na^2}{m}} = a\sqrt{\frac{n}{m}}$$

2. In the same way, when we put $BC = b$, $BE = y$, we find

$$y = b\sqrt{\frac{n}{m}}.$$

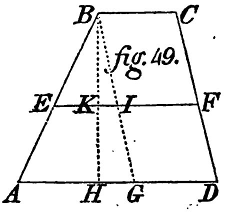
EXAM. From the triangle ABC it is required to cut off the fifth part by the parallel line DE; the line AB contains 739 parts of a certain scale: how many of these parts must be taken from B towards D, in order to determine the point D? Ans. $330\frac{1}{2}$ nearly.

Exam. 2. From a field, which is of the form of the triangle ABC, and which contains 14356 □′, it is required to cut off a piece containing 3958 □′ by a parallel line DE. When the side $AB = 573'$: what is the size of BD? Ans. 300′·867.

SECTION XLI.

Prob. *From a given quadrilateral with two parallel sides, to cut off, by a line parallel to these sides, a part consisting of a given area.*

Solut. Let $ABCD$ (*fig.* 49) be the trapezium, with two parallel sides AD, BC, from which trapezium, by a line EF, parallel to these two sides, it is required to cut off a part $BCFE$, whose area $= q$.



1. Draw the perpendicular BH, and BG parallel to CD, and let $AD = a$, $BC = b$, and the altitude $BA = h$. If we knew how to determine the point K, in which the lines BH, EF intersect each other, we could then draw the line of division. Let $\therefore BK = x$, and $EF = y$.

2. Since EI is parallel to AG, therefore

$$AG : EI = BG : BI = BH : BK$$

or
$$a - b : y - b = h : x$$

consequently
$$(a - b)\, x = (y - b)\, h$$

3. $$\text{Trapez. } BCFE = \tfrac{1}{2}(y + b)\, x = q$$

consequently
$$(y + b)\, x = 2q$$

4. Therefore the two equations 2 and 3, when solved, give:

$$y = \sqrt{\left[\frac{2q(a-b)}{h} + b^2\right]}$$

$$x = \frac{h}{a-b}\left[-b + \sqrt{\left(\frac{2q(a-b)}{h} + b^2\right)}\right]$$

EXAM. Let $a = 76'$, $b = 36'$, $h = 23'$, and $\therefore$ the area of the trapezium $= 1288\ \square'$; it is required to cut off from it a part containing $560\ \square'$: what is the length of the line of section EF, and its distance from BC?

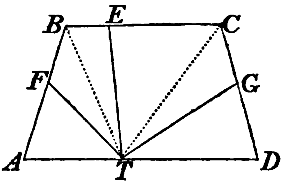
Ans. $EF = 56'\cdot954$, and $BK = 12\cdot048$.

SECTION XLII.

PROB. *To divide a trapezium with two parallel sides, in a given proportion, from a given point in one of its sides.*

SOLUT. Let $ABCD$ (*fig.* 50) be the trapezium, which from the point T is to be divided by a line TE in such a way, that the section $ABET$ is to the whole trapezium, as $n : m$. Let $AD = a$, $BC = b$, $AB = c$, $CD = d$, $AT = f$.

fig. 50.



1. Put $BE = x$; then, because the altitudes are equal, by § XXV, Remark,

$$\text{Trapez. } ABCD : \text{Trapez. } ABET = a + b : f + x;$$

but $\quad \text{Trapez. } ABCD : \text{Trapez. } ABET = m : n$

consequently $\quad a + b : f + x = m : n$

$$\therefore \quad x = \frac{n(a+b)}{m} - f.$$

2. If x in the course of the operation be found to be negative, this indicates, that the point E is not situated in BC, but in AB. In this case, let TF be the line of section, and $AF = y$. Draw BT; then

$$\text{Trapez. } ABCD : \triangle ATB = a + b : f$$

and $\quad \triangle ATB : \triangle ATF = c : y;$

consequently $\quad \text{Trapez. } ABCD : \triangle ATF = c(a+b) : fy.$

But $\quad \text{Trapez. } ABCD : \triangle ATF = m : n$

consequently $\quad c(a+b) : fy = m : n$

$$\therefore \quad y = \frac{nc(a+b)}{mf}.$$

3. If in 1 x be found greater than b, this indicates that the line of section must fall in CD. Let $\therefore$ TG be the line of section, and $DG = z$. Draw CT; then

$$\text{Trapez. } ABCD : \triangle CTD = a + b : a - f$$

and

$$\triangle CDT : \triangle GTD = d : z$$

consequently

$$\text{Trapez. } ABCD : \triangle GTD = d\,(a + b) : (a - f)\,z.$$

But according to the hypothesis,

$$\text{Trapez. } ABCD : ABCGT = m : n$$

and $\therefore$

$$\text{Trapez. } ABCD : \triangle GTD = m : m - n.$$

We have $\therefore$

$$d\,(a + b) : (a - f)\,z = m : m - n$$

and

$$z = \frac{(m - n)\,(a + b)\,d}{m\,(a - f)}$$

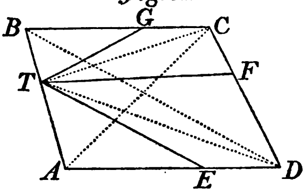
Exam. Let $a = 112'$, $b = 80'$, $c = 45'$, $d = 40'$, $f = 30'$. If it is required to cut off the third part from the trapezium, assume $BE = x = 34'$, and draw TE. If the tenth part is to be cut off, assume $AF = y = 28' \cdot 8$, and draw TF; but if $\frac{5}{8}$ths of the trapezium are to be cut off, assume $DG = z = 35' \cdot 112$, or thereabouts, and draw TG.

SECTION XLIII.

Prob. *To divide a trapezium having two parallel sides in a given proportion, from a given point not in the parallel sides.*

Solut. Let $ABCD$ (*fig.* 51) be the trapezium; AD, BC the parallel sides, and T the point from which the line of section is drawn.

fig. 51.



First calculate the triangles ATD, CTD, BTC, with reference to the trapezium; then from the magnitude of these triangles, and from the magnitude of the part to be cut off, we may easily judge whether

the point E in the line of section TE, falls in AD, CD, or BC. Let $\therefore$ $AD = a$, $BC = b$, $TA = c$, $TB = d$; and the area of the trapezium $= A$.

1. If we draw the lines AC, BD, then we find, by similar conclusions to those made in the foregoing §,

$$\triangle\, ATD = \frac{ac}{(a+b)(c+d)} . A$$

$$\triangle\, BTC = \frac{bd}{(a+b)(c+d)} . A$$

$$\triangle\, CTD = \text{Trapez. } ABCD - \triangle\, ATD - \triangle\, BTC$$

$$= \frac{ad+bc}{(a+b)(c+d)} . A$$

2. If it is required to cut off a given part from the trapezium, it is only necessary to divide one of these triangles, from its vertex T, in a given proportion, which, by § X, is done by dividing its base in this proportion. The following example will elucidate this.

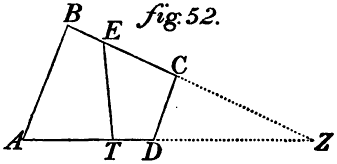
Exam. Let $a = 120'$, $b = 98'$, $c = 46'$, $d = 31'$; also $\triangle\, ATD = 0{\cdot}3288.A$, $\triangle\, BTC = 0{\cdot}1810.A$, $\triangle\, CTD = 0{\cdot}4902.A$. If it is required to cut off the fourth part of the trapezium, or $\frac{1}{4}\ A = 0{\cdot}25.A$, we must then divide AD in E, so that $AD : AE = 3288 : 2500$, and then, if we draw TE, TAE is the fourth part. If it is required to cut off $\frac{2}{3}$rds of the trapezium, or $\frac{2}{3}\ A = 0{\cdot}6666 . A$, we must in this case add a $\triangle\, DTF = 0{\cdot}3378$ to ATD, and consequently divide DC in F, in such a way, that $DC : DF = 4902 : 3378$; then $ATFD$ will be the part required. If we wish to cut off the $\frac{8}{9}$th part, or $\frac{8}{9}\ A = 0{\cdot}8888 . A$, we must, because $\triangle\, ATD + \triangle\, DTC = 0{\cdot}8190 . A$, add a $\triangle\, CTG = 0{\cdot}0698 . A$ to the quadrilateral $ATCD$, and consequently divide BC in G, so that $BC : CG = 1810 : 698$; then $ATGCD$ will be the part required.

SECTION XLIV.

Prob. *From a given point to divide any trapezium in a given proportion.*

Solut. Let $ABCD$ (*fig.* 52) be the given trapezium,

and T the point from which the line of division TE is so drawn that trapez. $ABCD$: trapez. $DCET = m : n$.



1. Produce the sides BC, AD till they meet in Z. Since the trapez. $ABCD$ is given, the lines AZ, BZ, CZ, DZ, may also be determined. Let $\therefore$ $AZ=a$, $BZ=b$, $DZ=c$, $CZ=d$. Since the point T is also given, let $ZT=f$. In order now to determine the point E, we put $ZE=x$.

2. By § XXVI, Cor. 1,

$$\triangle BZA : \triangle CZD = ab : cd$$

$\therefore$ $\triangle BZA - \triangle CZD : \triangle CZD = ab - cd : cd$
or trapez. $ABCD : \triangle CZD = ab - cd : cd$

3. In like manner

$$\triangle EZT : \triangle CZD = fx : cd$$

$\therefore$ $\triangle EZT - \triangle CZD : \triangle CZD = fx - cd : cd$
or trapez. $DCET : \triangle CZD = fx - cd : cd$.

4. From 2 and 3 we obtain
Trapez. $ABCD$: trapez. $DCET = ab - cd : fx - cd$,
But trapez. $ABCD$: trapez. $DCET = m : n$
consequently $ab - cd : fx - cd = m : n$
and
$$x = \frac{n(ab - cd)}{mf} + \frac{cd}{f}.$$

Exam. Let $a = 200'$, $b = 178'$, $c = 112'$, $d = 120'$, $f = 140'$, and it is required to cut off $\frac{3}{8}$ths of the trapezium. Assume $ZE = 155\frac{5}{14}$, and draw TE; then $DCET$ is the part required.

SECTION XLV.

Prob. *To divide a quadrilateral, which is already divided by a straight line into two other quadrilaterals, by another straight line, in such a way, that from each of the two quadrilaterals, into which the whole figure is divided, parts may be cut off containing given areas.*

Solut. The quadrilateral $ABCD$ (*fig.* 53), which is divided into the quadrilaterals $EBCF$, $EADF$, by the line EF, is required to be divided by the line IL, so that the quadrilaterals, $BEKI$, $EALK$, may have given areas.

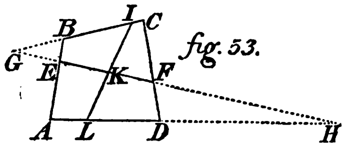


fig. 53.

1. Produce the sides BC, AD, till they meet EF produced in G and H. Since the trapez. $ABCD$, and one of its lines EF, are given, we can ∴ assume the areas of the triangles BEG, AEH, also the angles BGE, AHE, and the line GH as known. Now, since the areas of the trapeziums $BEKI$, $EALK$ are known, consequently also the areas of the triangles GKI, LKH are known. Put therefore $\triangle\, GKI = p$, $\triangle\, LKH = q$, $\angle\, IGK = \alpha$, $\angle\, KHL = \beta$, and $GH = a$.

2. If the line GK, and the angle GKI are known, we can draw the line of division IL. Put ∴ $GK = x$, $\angle\, GKI = \phi$.

3. By § XXVII,

$$\triangle\, GKI = \frac{x^2 \, Sin.\, \alpha \, Sin.\, \phi}{2\, Sin.\, (\alpha + \phi)},$$

$$\triangle\, LKH = \frac{(a - x)^2 \, Sin.\, \beta \, Sin.\, \phi}{2\, Sin.\, (\beta + \phi)}.$$

Now since $\triangle\, GKI = p$, $\triangle\, LKH = q$, we have the two equations,

$$x^2 \, Sin. \, \alpha \, Sin. \, \phi = 2 \, p \, Sin. \, (\alpha + \phi)$$
$$(a-x)^2 \, Sin. \, \beta \, Sin. \, \phi = 2 \, q \, Sin. \, (\beta + \phi).$$

4. Expand $Sin. \, (\alpha + \phi)$, $Sin. \, (\beta + \phi)$ divide the first equation by $Sin. \, \alpha \; Sin. \, \phi$, and the second by $Sin. \, \beta \; Sin. \, \phi$, and put $Cot. \, \alpha$, $Cot. \, \beta$, $Cot. \, \phi$, for $\frac{Cos. \, \alpha}{Sin. \, \alpha}$, $\frac{Cos. \, \beta}{Sin. \, \beta}$, $\frac{Cos. \, \phi}{Sin. \, \phi}$. By these means the foregoing equations are transformed into the following ones:

$$x^2 = 2 \, p \, (Cot. \, \phi + Cot. \, \alpha)$$
$$(a-x)^2 = 2 \, q \, (Cot. \, \phi + Cot. \, \beta).$$

5. Hence by eliminating $Cot. \, \phi$, we obtain

$$\frac{(a-x)^2}{2 \, q} - \frac{x^2}{2p} = Cot. \, \beta - Cot. \, a$$

or $\dfrac{(a-x)^2}{2q} - \dfrac{x^2}{2p} = \dfrac{Sin. \, (\alpha - \beta)}{Sin. \, \alpha \, Sin. \, \beta}$,

or $x^2 - \dfrac{2ap}{p-q} \, x = \dfrac{2pq \, Sin. \, (\alpha - \beta)}{(p-q) \, Sin. \, \alpha \, Sin. \, \beta} - \dfrac{a^2 p}{p-q}$

6. The solution of this equation gives

$$x = \frac{ap}{p-q} \pm \sqrt{\left[\frac{a^2 pq}{(p-q)^2} + \frac{2pq \, Sin. \, (\alpha - \beta)}{(p-q) \, Sin. \, \alpha \, Sin. \, \beta}\right]},$$

or $x = \dfrac{a}{p-q}\left[p \pm \sqrt{\left(pq + \dfrac{2pq \, (p-q) \, Sin. \, (\alpha - \beta)}{a^2 \, Sin. \, \alpha \, Sin. \, \beta}\right)}\right]$.

7. Having found x, ϕ is also known, for

$$Cot. \, \phi = \frac{x^2}{2p} - Cot. \, \alpha.$$

Cor. If $\alpha = \beta$, then $Sin. \, (\alpha - \beta) = 0$, and we obtain

$$x = \frac{a}{p-q} \, [p \pm \sqrt{pq}].$$

If we put $p = q$, from the formula in 6, we obtain

$$x = \frac{a}{p-p}[p \pm \sqrt{p^2}],$$

consequently, either $p = \infty$, or $x = \frac{0}{0}$. The first of these two values cannot be used here; the second is indeterminate. But for this particular case, we have from 4, the two following equations:

$$x^2 = 2p\,(Cot.\,\phi + Cot.\,\alpha)$$
$$(a - x)^2 = 2p\,(Cot.\,\phi + Cot.\,\beta).$$

If the second be subtracted from the first, we then obtain an equation of the first degree only, viz.

$$2ax - a^2 = 2p\,(Cot.\,\alpha - Cot.\,\beta) = \frac{2p\,Sin.\,(\alpha-\beta)}{Sin.\,\alpha\,Sin.\,\beta}$$

and $\therefore\ x = \dfrac{p\,Sin.\,(\alpha-\beta)}{a\,Sin.\,\alpha\,Sin.\,\beta} + \frac{1}{2}a.$

If besides in this case $\beta = \alpha$, we obtain $x = \frac{1}{2}a$, which is also easily inferred from the figure, because by reason of the equal angles, and the equal areas of the triangles GKI, LKH, it necessarily follows that $GK = KH$.

Exam. A square field $ABCD$, which consists of two parts, the part $EBCF$ of pasture, and the part $EADF$ of arable land, is required to be divided by a line IL, so that the part $BEKI$, which is cut off from the pasture-land, has an area of $2600\ \square^0$, and the part $EALK$, which is cut off from the arable land, has an area of $2900\ \square^0$.

Let $CGF = \alpha = 40^0.\ 40'$, $AHE = \beta = 18^0.\ 26'$, $GH = a = 228^0$, $\triangle\,BEG = 480\ \square^0$, $\triangle\,AEH = 6488\ \square^0$; $\therefore$ $\triangle\,GKI = p = 3080\ \square^0$, $\triangle\,LKH = q = 3588\ \square^0$. The calculation is effected in the following way:

$$\frac{2pq\,(q-p)\,Sin.\,(\alpha-\beta)}{a^2\,Sin.\,\alpha\,Sin\,\beta} = u:$$

then

$$x = \frac{a}{q-p}[\pm\sqrt{(pq-u)} - p].$$

The calculation by means of logarithms gives

$$\begin{aligned} log.\, u &= [log.\, 2p + log.\, q + log.\, (q-p) + log.\, Sin.\, (\alpha-\beta)] - \\ &\quad [2\, log.\, a + log.\, Sin.\, \alpha + log.\, Sin.\, \beta] \\ &= [log. 6160 + log. 3588 + log. 508 + log. Sin. 22^0 14'] - \\ &\quad [2\, log.\, 228 + log.\, Sin.\, 40^0\, 41' + log.\, Sin.\, 18^0\, 26'] \\ &= 9{\cdot}6282243 - 4{\cdot}0298522 = 5{\cdot}5983721, \end{aligned}$$

$\therefore$

$$u = 396617{\cdot}63\,;\ \sqrt{(pq - u)} = 3264{\cdot}11$$

and $x = \dfrac{228}{508}[3264{\cdot}11 - 3080] = 82{\cdot}63.$

Of the two values found for x, only the first can be used here, because the second is negative.

Hence we further obtain

$$Cot.\, \phi = \frac{x^2}{2q} - Cot.\, \alpha = 1{\cdot}10839 - 1{\cdot}16397 = -0{\cdot}05558,$$

and $\therefore$ $\phi = 93^0\, 10'.$

If $\therefore$ we make $GK = 82^0.\, 63'$, and through K draw the line IL, forming an angle $GKI = 93^0.\, 10'$, then the required part is cut off.

REMARK. The problem here solved is of the greatest importance to the practical surveyor. A similar problem is to be found in Lambert's learned German Correspondence, published by J. Bernoulli, 2nd vol. (1782), p. 412; also in Von Tempelhof's Supplement to Clairault's Rudiments of Algebra, (Second Edition, 1797), p. 225.

V. GEOMETRICAL DETERMINATION OF HEIGHTS AND DISTANCES.*

SECTION XLVI.

PROB. *To determine the distance between two objects, when there is an obstacle between them, as a sea, a morass, or mountain, the distance between them not being known, under the supposition that there is a station, from which we can measure the distance to these objects in a straight line.*

First Solution.

1. Let A and B (*fig.* 54) be the two objects, whose distance from each other is required to be found. Take a station C, from which the distances to A and B may be measured; measure the distances CA, CB, and from these last backwards measure off the same spaces towards D and E, make $CD=CA$, $CE=CB$. If, after this, we measure the distance DE, so that $AB = DE$, or, make $CD'=CA$, $CE'=CB$, then also $AB=D'E'$.

2. If the distances CA, CB are very great, and if it be impracticable to measure them backwards by reason of impediments, take merely an equal part of the two distances, the half, the third, fourth, or in general the nth part; measure off these parts backwards from C towards a, b, or forwards from C towards α, β; then ab, or $\alpha\beta$, is the same part of the

* The problems in this chapter must properly be considered merely as geometrical exercises (as appears from the mode of treatment already adopted), and consequently are not determined for surveyors only.

unknown distance AB, as Ca, or $C\alpha$ is of CA, and Cb, or $C\beta$ is of CB.

The reason of this is easily seen.

Second Solution.

Again, A and B, (*fig.* 55) are the two objects, and C the chosen station. In the direction CB, make the distance $CD = CA$, and if it be practicable, measure AD. Let $AC = CD = a$, $BC = b$, $AD = c$, $BD = b - a = d$: then in the triangle ACB,

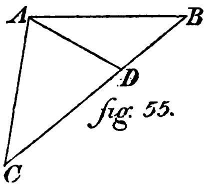


fig. 55.

$$AB^2 = a^2 + b^2 - 2ab \text{ Cos. } C,$$

and in the triangle ACD,

$$c^2 = 2a^2 - 2a^2 \text{ Cos. } C,$$

because $AC = CD = a$. If we subtract the value of Cos. C from the second equation, and substitute it in the first, we then obtain

$$AB^2 = a^2 + b^2 - 2ab + \frac{bc^2}{a}$$

$$= (a - b)^2 + \frac{bc^2}{a}$$

$$= d^2 + \frac{bc^2}{a},$$

and $\therefore$ $AB = \sqrt{\left(d^2 + \frac{bc^2}{a}\right)}$.

Exam. When $a = 50'$, $b = 76'$, $c = 32'$, $AB = 47' \cdot 249$.

Third Solution.

If in the directions CA, CB (*fig.* 56), there are two points D, E, whose distance DE can be measured, then, when the distances CA, CB, CD, CE are also measured, the distance of the objects A, B may in like manner be determined. Let $CA = a$, $CB = b$, $CD = c$, $CE = d$, $DE = e$; then in the triangle ACB,

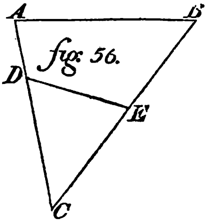


fig. 56.

$$AB^2 = a^2 + b^2 - 2ab \text{ Cos. } C,$$

and in the triangle DCE,

$$e^2 = c^2 = d^2 - 2cd\ Cos.\ C,$$

and when we substitute in the first equation the value of $Cos.\ C$ from the second equation

$$AB = \sqrt{[a^2 + b^2 - \frac{ab}{cd}(c^2 + d^2 - e^2)]}$$

Exam. When $a = 30^0$, $b=35^0$, $c=20^0$, $d=15^0$, $e=13^0$, $AB=23^0$.

Fourth Solution.

If we are provided with instruments for measuring angles, it will only be necessary (*fig.* 56) to measure the angle ACB, and the distances CA, CB. Let $CB = a$, $CA = b$, $\angle\ ACB = \alpha$; then

$$AB = \sqrt{(a^2 + b^2 - 2ab\ Cos.\ \alpha)}.$$

But we can also first determine the angles CAB, CBA. For since $CAB + CBA = 180^0 - \alpha$, $\therefore$, by means of the proportion,

$$a + b : a - b = Tan. \frac{CAB + CBA}{2} : Tan. \frac{CAB - CBA}{2}$$

$$= Tan. \frac{180^0 - \alpha}{2} : Tan. \frac{CAB - CBA}{2}$$

it will merely be necessary to find the difference between the two angles CAB, CBA. If the angles CAB, CBA are determined, we then have

$$AB = \frac{b\ Sin.\ \alpha}{Sin.\ CAB} = \frac{a\ Sin.\ \alpha}{Sin.\ CBA}.$$

Cor. If $AC = BC$, we then find

$$AB = \sqrt{(2a^2 - 2a^2\ Cos.\ \alpha)} = a\sqrt{2\ (1 - Cos.\ \alpha)} = 2\ a\ Sin.\ \tfrac{1}{2}\alpha$$

If ACB be a right angle, then $Cos.\ \alpha = 0$, and $\therefore$

$$AB = \sqrt{(a^2 + b^2)}.$$

If $ACB = 45^0$, then because $Cos.\ 45^0. = \sqrt{\tfrac{1}{2}}$,

$$AB = \sqrt{(a^2 + b^2 - ab\sqrt{2})}$$

Exam. 1. When $a = 168'$, $b = 102'$, $\alpha = 49^\circ.25'$, $AB = 127'\cdot 797$.

Exam. 2. When $a = 189'$, $b = 114\frac{3}{4}'$, $\alpha = 107^\circ.48'$, $AB = 249'\cdot 295$.

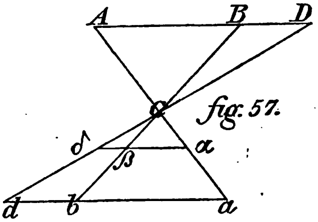
Exam. 3. When $a = b = 250'$, $\alpha = 43^\circ.50'$, $AB = 186'\cdot 629$.

SECTION XLVII.

Prob. *To determine the distance between two objects, when only one of them is accessible.*

First Solution.

Suppose A (*fig.* 57) is an inaccessible object, and B another, whose distance from A it is required to find.



1. In AB produced, assume any point D, and find a station C, from which both B and D are accessible; measure the distances CB, CD, and measure these off backwards on CB, CD, produced; then make $Cd = CD$, and $Cb = CB$; then proceed in the direction db till a point a is arrived at, which lies in a straight line with A and C. If we now measure the distance ba, we then also have the distance AB, because $AB = ab$.

2. If there be not sufficient space to measure off the whole of the distances CB, CD backwards, it will only be necessary to take equal parts of them, $C\delta = \frac{1}{n} CD$, $C\beta = \frac{1}{n} CB$, then to proceed in the direction $\delta\beta$, till we arrive at a point α, which lies in a straight line with A, C. If after this we measure $\beta\alpha$, we shall then find AB from the proportion

$$AB : \beta\alpha = CB : C\beta = CD : C\delta.$$

The reason of this method is easily discovered.

Second Solution.

If we can measure from C towards A and B (*fig.* 57), also from B towards A and C, and likewise the angles ACB, ABC, and besides these the distance BC, then

$$AB = \frac{BC \text{ Sin. } ACB}{\text{Sin. } (ACB + ABC)}.$$

Exam. When $BC = 738'$, $ACB = 24^\circ.\ 16'.\ 13''$, $ABC = 31^\circ.\ 5$, $AB = 368'{\cdot}734$.

Third Solution.

Let AB (*fig.* 58) be the distance to be measured, from which the point B is only accessible; C a point beyond it, and D a point in AB produced.

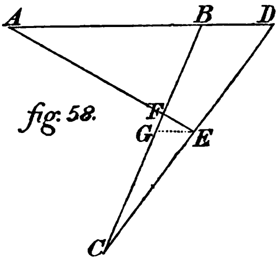


fig: 58.

1. In CD take any point E, and proceed in the direction EA, till a point F is arrived at, which is in a straight line with B, C; then measure the distances BD, BF, FC, CE, ED: from hence the distance AB may be determined. Let $BD = a$, $BF = b$, $CF = c$, $DE = d$, $EC = e$.

2. Draw EG parallel to BD; then because the triangles CGE, CBD are similar,

$$CD : CE = BD : GE$$

or $$d + e : e = a : GE$$

and $$CD : CE = BC : CG$$

or $$d + e : e = b + c : CG,$$

consequently $GE = \dfrac{ae}{d + e}$, $CG = \dfrac{(b + c)e}{d + e}$

$$FG = CF - CG = \frac{cd - be}{d + e}$$

3. The triangles EFG, AFB, are in like manner similar; we $\therefore$ have

$$FG : GE = BF : AB,$$

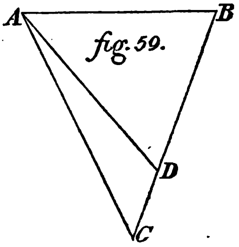
or $\frac{cd - be}{d + e} : \frac{ae}{d + e} = b : AB$,

consequently $AB = \frac{abe}{cd - be}$.

Exam. When $a = 250'$, $b = 76'$, $c = 132'$, $d = 140'$, $e = 80'$, $AB = 122\frac{18}{31}'$.

Fourth Solution.

Again, let AB (*fig.* 59) be a distance which is accessible only in B, but which cannot be produced in any direction.



1. Take two stations C, D, which are in a straight line with B, from which also both A and B may be seen; measure the distances CD, CB, and the angles ACB, ADB. Let $BC = a$, $CD = b$, $ADB = \alpha$, $ACB = \beta$.

2. In the triangle ACD

$$Sin.\ CAD : Sin.\ ADC = CD : AC,$$

or $Sin.\ (\alpha - \beta) : Sin.\ \alpha = b : AC$,

$\therefore AC = \frac{b\ Sin.\ \alpha}{Sin.\ (\alpha - \beta)}$.

3. But in the triangle ACB,

$$AB^2 = BC^2 + AC^2 - 2BC\ .\ AC\ .\ Cos.\ \beta$$

or $$AB^2 = a^2 + \left(\frac{b\ Sin.\ \alpha}{Sin.\ (\alpha - \beta)}\right)^2 - \frac{2\ ab\ Sin.\ \alpha\ Cos.\ \beta}{Sin.\ (\alpha - \beta)},$$

consequently $AB = \sqrt{\left[a^2 + \left(\frac{b\ Sin.\ \alpha}{Sin.\ (\alpha - \beta)}\right)^2 - \frac{2ab Sin.\alpha Cos.\beta}{Sin.\ (\alpha - \beta)}\right]}$

Exam. Let $a = 500'$, $b = 67'$, $\alpha = 38^0.\ 7'$, $\beta = 21^0.\ 43'$.

Here

$$a^2 = 250000$$

$$\left(\frac{b \ Sin. \ \alpha}{Sin. \ (\alpha - \beta)}\right)^2 = 21455{\cdot}72$$

$$\frac{2 \ ab \ Sin. \ \alpha \ Cos. \ \beta}{Sin. \ (\alpha - \beta)} = 186081{\cdot}46$$

consequently $AB = \sqrt{135374{\cdot}26} = 367'{\cdot}932.$

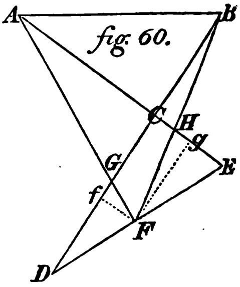
SECTION XLVIII.

Prob. *To find the distance of two objects from one another, when neither of them is accessible.*

First Solution.

Let AB (*fig.* 60) be the distance to be measured.

1. Take a position C, extend the directions AC, BC, indefinitely towards D and E, and bisect DE in F. Measure from F towards A and B, and determine the points G, H, in which the sight-lines FA, FB, cut the lines CD, CE. Then measure the three sides of the triangle DCE, and the distances CG, CH. Let $CD = a$, $CE = b$, $DE = c$, $CG = d$, $CH = e$.



2. Draw Ff parallel to CE; then, because DE is bisected in F, $Cf = \frac{1}{2}a$, $Gf = \frac{1}{2}a - d$, and because the triangles AGC, FGf are similar,

$$Gf : Ff = CG : AC$$

or $$\tfrac{1}{2}a - d : \tfrac{1}{2}b = d : AC$$

consequently $$AC = \frac{bd}{a - 2d}.$$

3. In like manner, when Fg is parallel to CD, we find

$$BC = \frac{ae}{b - 2e}.$$

4. After the lines AC, BC have been found, it will only be necessary, by the first solution, § XLVI, to measure these lines themselves, or proportional parts of them, backwards on the lines produced; then the distance AB will be deduced from hence.

COR. If we wish to determine AB by arithmetic, it may be done in the following way.

Since the three sides of the triangle DEC are given, we have

$$Cos.\ DCE = \frac{a^2 + b^2 - c^2}{2ab}.$$

But in the triangle ACB, we have

$$AB^2 = AC^2 + BC^2 - 2\,AC\,.\,BC\,.\,Cos.\ ACB.$$

If ∴ for AC, BC, and $Cos.\ ACB = Cos.\ DCE$, we substitute their values already found, we then obtain

$$AB = \surd\left[\frac{b^2d^2}{(a-2d)^2} + \frac{a^2e^2}{(b-2e)^2} - \frac{de(a^2+b^2-c^2)}{(a-2d)\,(b-2e)}\right];$$

to which expression we can also give the following form, which is more convenient for calculation by logarithms:

$$AB = \surd\left[\left(\frac{bd}{a-2d} - \frac{ae}{b-2e}\right)^2 - \frac{de\,(a-b+c)\,(a-b-c)}{(a-2d)\,(b-2e)}\right].$$

EXAM. When $a = 156'$, $b = 98'$, $c = 187'$, $d = 63'$, $e = 34'$, $AB = 275'\cdot791$.

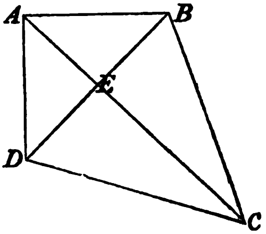
Second Solution.

Let AB (*fig.* 61) be the inaccessible distance.

1. Take any point whatever, C, measure towards A and B, and by these means determine the angle ACB; make

$ACD = ACB$, and proceed in the direction CD, till we find a point D, where the angle $BDC = 90^\circ - BCA$, and measure the distance CD, and the angle ADC. Let $BCA = ACD = \alpha$, $ADC = \beta$, $CD = a$.

fig. 61.



2. Since $ACD = \alpha$, and $BDC = 90^\circ - \alpha$, CED is a right angle, consequently $\triangle\ CED$ is similar to $\triangle\ BEC$, and $\therefore BC = CD$. Consequently also $\triangle\ ACD$ is similar to $\triangle\ ACB$, and $\therefore AD = AB$.

3. But in the triangle ACD

$$\text{Sin. } DAC : \text{Sin. } DCA = CD : AD$$

or $$\text{Sin. } (\alpha + \beta) : \text{Sin. } \alpha = a : AD\ (= AB),$$

consequently $$AB = \frac{a \text{ Sin. } \alpha}{\text{Sin. } (\alpha + \beta)}.$$

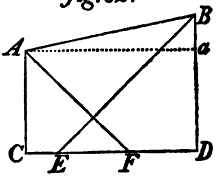
Exam. When $\alpha = 31^\circ.\ 5'$, $\beta = 113^\circ.\ 17'$, $a = 567'$, $AB = 502'{\cdot}463$.

Third Solution.

Let AB (*fig.* 62) be the inaccessible distance.

1. Take any position C, make $ACD = 90^\circ$, proceed in the direction CD, till the point D is arrived at, likewise $BDC = 90^\circ$, and in the line CD find two points E, F, so situated, that $AFC = BED = 45^\circ$.

fig. 62.



2. Having determined these points, measure the distances CD, CF, DE. Let $CD = a$, $CF = b$, $DE = c$. Draw Aa parallel to CD.

3. Then $aD = AC = CF = b$, and $BD = DE = c$; consequently $Ba = c - b$.

4. But in the right-angled triangle ABa, $AB^2 = Aa^2 + Ba^2 = CD^2 + Ba^2$; $\therefore$

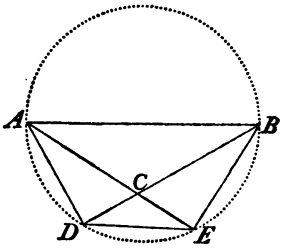
$$AB = \sqrt{[a^2 + (c-b)^2]}$$

Fourth Solution.

Let AB (*fig.* 63) be the distance to be measured.

1. Take any position C, where ACB is an obtuse angle, measure this angle in measuring towards A and B, then its adjacent angle ACD is also known. Now, on the sight-lines AC, BC, fix on two points D, E, in such a position, that $ADB = AEB = 90^0$, and measure the distance DE, then from hence AB may be determined. Let $DE = a$, $ACD = \alpha$.

fig. 63.



2. Since ADB, AEB, by the construction are right angles, the points D, E are in a circle, whose radius is AB. If we suppose this circle actually described, then DAE is an angle at the circumference, and DE is the chord of the arc, upon which it stands; consequently, from known trigonometrical principles

$$DE = a = AB \ Sin. \ DAE.$$

But $Sin. \ DAE = Cos. \ \alpha$, (because ADC is a right angle), $\therefore a = AB \ Cos. \ \alpha$, and

$$AB = \frac{a}{Cos. \ \alpha}.$$

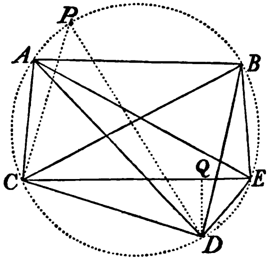
Exam. When $a = 563'\cdot7$, $\alpha = 42^0. \ 19'\cdot7''$, $AB = 762' \ 362$.

Fifth Solution.

Let AB (*fig.* 64) be the distance to be measured.

1. Fix upon three stations C, D, E, so that the three angles ACB, ADB, AEB, which, in measuring towards A and B, include the sight-lines, are equal to one another; let each of these angles $= \alpha$. Then measure the three sides of the triangle CDE, and from these we can determine the distance AB. Let $CD = a$, $DE = b$, $CE = c$.

fig. 64.



2. Upon AB describe an arc, which includes the given angle α, and complete the circle; then this circle will be given by the three points C, D, E. Draw the diameter DP, the line PC, and the perpendicular DQ.

3. The triangles DCP, DQE are similar; for $DQE = DCP = R$, and $DEC = DPC$; consequently

$$DQ : DE = CD : DP,$$

or, because DQ, is the altitude of the triangle CDE, $= \frac{2 \triangle CDE}{c}$,

$$\frac{2 \triangle CDE}{c} : b = a : DP,$$

and $\therefore$
$$DP = \frac{abc}{2 \triangle CDE}.$$

4. If in this expression we substitute for the triangle CDE its value from § XXIX, we then obtain

$$DP = \frac{2abc}{\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}}.$$

5. When the diameter of the circle is found, it is easy to determine the chord AB. For since $AB = DP \text{ Sin. } \alpha$, when

for DP its value is substituted,

$$AB = \frac{2\,abc\,Sin.\,\alpha}{\sqrt{(a+b+c)\,(a+b-c)\,(a+c-b)\,(b+c-a)}},$$

which expression readily admits of being calculated by logarithms.

Exam. When $a = 197'$, $b = 113'$, $c = 235'$, $\alpha = 56^{0}.\ 29'$, $AB = 196'\cdot536$.

Cor. From 4 it appears at once, that when $a, b, c,$ are the three sides of a triangle, the radius of the circle described about it $=$

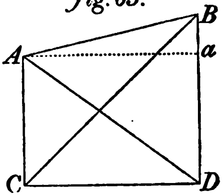
$$\frac{abc}{\sqrt{(a+b+c)\,(a+b-c)\,(a+c-b)\,(b+c-a)}}.$$

Sixth Solution.

Let AB (*fig.* 65) be the distance to be measured.

1. Take any position C, make the right angle ACD, and proceed in the direction CD, as far as the point D, where also BDC is a right angle; measure from C towards B and D, likewise from D towards A and C, and determine by these means the angles BCD, ADC; measure also the line CD. Let $BCD = \alpha$, $ADC = \beta$, $CD = a$.

fig. 65.



2. From the right-angled triangles ACD, BDC, we obtain,

$$BD = a\ Tan.\ \alpha,\ AC = a\ Tan.\ \beta.$$

If Aa is drawn parallel to CD; then

$$Aa = a,\ Ba = a\,(Tan.\ \alpha - Tan.\ \beta).$$

3. Therefore in the right-angled triangle AaB

$$AB^2 = a^2 + a^2\,(Tan.\ \alpha - Tan.\ \beta)^2$$

and consequently $AB = a\sqrt{[1 + (Tan.\ \alpha - Tan.\ \beta)^2]}$.

4. In order the more easily to calculate the expression found for AB, put $Tan.\ \alpha - Tan.\ \beta = Tan\ \phi$, $\therefore$ find an angle ϕ such, that its tangent is equal to the difference of the tangents of the two angles α, β. Having found this angle, then

$$AB = a\sqrt{(1 + Tan.^2\ \phi)} = a\ Sec.\ \phi$$

EXAM. Let $a = 1375'$, $\alpha = 65^\circ.\ 17'$, $\beta = 39^\circ.\ 48'$. Here

$$Tan.\ \phi = Tan.\ \alpha - Tan.\ \beta = 2{\cdot}1724911 - 0{\cdot}8331686$$
$$= 1{\cdot}3393225$$

consequently $\phi = 53^\circ.\ 15'\ 11''$.

Hence we obtain,

$$log.\ Sec.\ \phi = log.\ Sec.\ 53^\circ.\ 15'.\ 11'' = 0{\cdot}2230941$$
$$log.\ a\ Sec.\ \phi = 3{\cdot}3613968$$
$$\therefore AB = 2298'{\cdot}247.$$

Seventh Solution.

Again let AB (*fig.* 65) be the distance to be measured.

1. Take any two positions C, D; measure the angles ACD, BCD, BDC, ADC also the line of vision CD. Let $CD = a$, $ACD = \alpha$, $BCD = \beta$, $BDC = \gamma$, $ADC = \delta$; then in the triangle ACD, $\angle\ CAD = 180^\circ - \alpha - \delta$, and in the triangle BCD, $\angle\ CBD = 180^\circ - \gamma - \beta$. Since $\therefore$ these last angles are also known, for shortness' sake, put $CAD = A$, $CBD = B$.

2. In the triangle CAD we $\therefore$ have

$$Sin.\ A : Sin.\ \delta = a : AC,$$

and in the triangle CBD

$$Sin.\ B : Sin.\ \gamma = a : BC,$$

consequently $AC = \dfrac{a\ Sin.\ \delta}{Sin.\ A}$, $BC = \dfrac{a\ Sin.\ \gamma}{Sin.\ B}$.

3. Now, since in the triangle ACB, both the two sides AC, BC, and also the angle included by them $ACB = \alpha - \beta$,

are known, we $\therefore$ obtain

$AB =$

$$a\sqrt{\left[\left(\frac{Sin.\ \delta}{Sin. A}\right)^2 + \left(\frac{Sin.\ \gamma}{Sin.\ B}\right)^2 - \frac{2\ Sin.\ \delta\ Sin.\ \gamma\ Cos.\ (\alpha-\beta)}{Sin.\ A\ Sin.\ B}\right]}.$$

In this expression we must calculate each part of the magnitudes under the radical sign separately; but the calculation will be essentially shortened, by using in the third part the logarithms of $\frac{Sin.\ \delta}{Sin. A}$ and $\frac{Sin.\ \gamma}{Sin.\ B}$, which must be calculated for the two first parts.

Exam. Let $\alpha = 110^0$, $\beta = 37^0.\ 40'$, $\gamma = 117^0.\ 30'$, $\delta = 38^0.\ 20'$, $a = 750'$. Here $A = 31^0.\ 40'$, $B = 24^0.\ 50'$,

$$\left(\frac{Sin.\ \delta}{Sin. A}\right)^2 = 1{\cdot}395832$$

$$\left(\frac{Sin.\ \gamma}{Sin.\ B}\right)^2 = 4{\cdot}460672$$

$$\frac{2\ Sin.\ \delta\ Sin.\ \gamma\ Cos.\ (\alpha-\beta)}{Sin.\ A\ Sin.\ B} = 1{\cdot}514520;$$

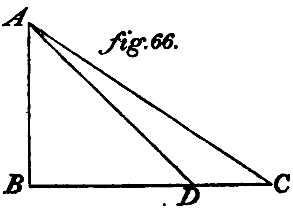
consequently $AB = 750\sqrt{4{\cdot}341984} = 1562'{\cdot}80$.

SECTION XLIX.

Prob. *To find the altitude of an object, for instance of a tower, when its base lies in the same horizontal plane with a chosen or given station, under the supposition, that the distance from this station to the object can be measured.*

First Solution.

Let AB (*fig.* 66) be the altitude of the object to be measured, or, more properly, the vertical line, which is drawn from the highest point of the object to its lowest, in the horizontal plane; let C be the chosen or given station, and $\therefore$, according to the hypothesis, BC is a horizontal line.



Measure CB, and the angle of elevation ACB. Let $CB = a$, $ACB = \alpha$; then in the right-angled triangle ABC,

$$AB = a \text{ Tan. } \alpha$$

Exam. When $a = 367'$, $\alpha = 32^{\circ}.\ 17'.23''$, $AB = 231'\cdot 915$.

Second Solution.

If the distance from C to B cannot be measured in a straight line, measure only a part of the line CB, say CD; also in C, D, the angles of elevation ACB, ADB. Let $CD = a$, $ACB = \alpha$, $ADB = \beta$.

In the triangle ADC, if the side CD, and the angles ACD, DAC $(= \beta - \alpha)$ be given; therefore

$$AD = \frac{a \text{ Sin. } \alpha}{\text{Sin. } (\beta - \alpha)}.$$

In the right-angled triangle ABD we $\therefore$ have the side AD, and the angle ADB; consequently

$$AB = AD \text{ Sin. } \beta = \frac{a \text{ Sin. } \alpha \text{ Sin. } \beta}{\text{Sin. } (\beta - \alpha)}.$$

Also BD, and consequently BC, may be determined; for

$$BD = AD \text{ Cos. } \beta = \frac{a \text{ Sin. } \alpha \text{ Cos. } \beta}{\text{Sin. } (\beta - \alpha)}.$$

Exam. When $a = 967'$, $\alpha = 7^{\circ}.\ 5'.\ 13''$, $\beta = 16^{\circ}.\ 43'.\ 5''$, $AB = 205'\cdot 131$, and $BD = 682'\cdot 955$.

SECTION L.

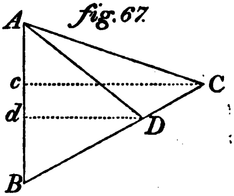
Prob. *To find the altitude of an object, when its lowest point is not in the same horizontal plane with the given station; on the supposition, that the distance from this station to the object can be measured.*

First Solution.

Let AB (*fig.* 67) be the altitude to be measured, and C the given station.

1. Arrange the telescope of the protractor to the horizontal direction Cc, then direct it towards A and B, and determine by these means the angle of elevation ACc, and the angle of depression BCc. If now the side CB is measured, then AB may be determined.

fig. 67.



2. For let $ACc = \alpha$, $BCc = \beta$; then $ACB = \alpha + \beta$, $BAC = 90^\circ - \alpha$; consequently in the triangle BAC, one side and two angles are given; $\therefore$, because $Sin.\ (90^\circ - \alpha) = Cos.\ \alpha$,

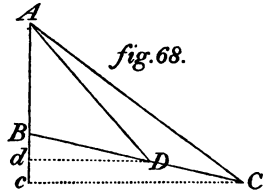
$$AB = \frac{a\ Sin.\ (\alpha + \beta)}{Cos.\ \alpha}$$

Cor. If, as in *fig.* 68, C is lower than B, it will merely be necessary to assume the angle $BCc = \beta$ to be negative, and we then obtain

$$AB = \frac{a\ Sin.\ (\alpha - \beta)}{Cos.\ \alpha}.$$

which may also be very easily proved from the figure itself.

fig. 68.



Exam. Let C be higher than B, and $\alpha = 19^\circ.\ 7'$, $\beta = 25^\circ.\ 13'$, $a = 1352'{\cdot}7$; then $AB = 1000'{\cdot}509$.

Second Solution.

1. If the distance from C to B cannot be measured, from C measure a part CD only; determine at C the angle of elevation ACc and angle of depression BCc, also at D the angle of elevation ADd. It is not necessary to measure the angle BDd, because $BDd = BCc$.

2. Let $ACc = \alpha$, $BCc = \beta$, $ADd = \gamma$; then $ACB = \alpha + \beta$, $ADB = \gamma + \beta$; and $\therefore$ $CAD = ADB - ACB = \gamma - \alpha$. Consequently in the triangle CAD all the angles and the side CD are known; consequently

$$AD = \frac{a \, Sin. \, (\alpha + \beta)}{Sin. \, (\gamma - \beta)}.$$

3. Now in the triangle *BAD*, the angles *ADB*, *ABD* ($= 90^0 - \beta$), and the side *AD* are known; consequently also *AB*. Thus

$$AB = \frac{AD \, Sin. \, (\gamma + \beta)}{Cos. \, \beta} = \frac{a \, Sin. \, (\alpha + \beta) \, Sin. \, (\gamma + \beta)}{Sin. \, (\gamma - \alpha) \, Cos. \, \beta}.$$

Cor. In *fig.* 67 it was assumed, that *C* was higher than *B*. But if (*fig.* 68) *C* be lower than *B*, it will only be necessary in the expression found for *AB*, to substitute $-\beta$ for β; we then get, because $Cos. - \beta = Cos. \, \beta$,

$$AB = \frac{a \, Sin. \, (\alpha - \beta) \, Sin. \, (\gamma - \beta)}{Sin. \, (\gamma - \alpha) \, Cos. \, \beta};$$

which may also be very easily proved immediately from the figure itself.

Exam. In *fig.* 68, let $\alpha = 29^0$, $\beta = 17^0. \, 6'$, $\gamma = 32^0. \, 49' \cdot 8''$, $a = 1152'$; then $AB = 1010' \cdot 953$.

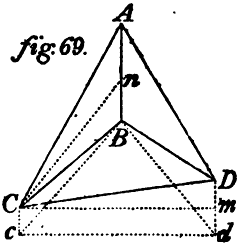
SECTION LI.

Prob. *To find the altitude of an object, when it is impracticable, from the chosen or given station in the direction of the object, to measure either forward or backward.*

First Solution.

Let *AB* (*fig.* 69) be the altitude to be measured, and *C* the given station.

1. Measure any horizontal station $CD = a$. Now, if this line be in the same horizontal plane *BCD*, as the lowest point *B* of the altitude *AB*; measure the horizontal angles *BCD*, *BDC*, also one of the angles of elevation *ACB*, *ADB*, viz. *ACB*. Let $BCD = \alpha$, $BDC = \beta$, $ACB = \gamma$.



2. Then in the triangle CBD there are two angles BCD, BDC, and the side CD given; consequently

$$CB = \frac{a \ Sin. \ \beta}{Sin. \ (\alpha + \beta)}.$$

3. But in the right-angled triangle ABC, having the right angle at B,

$$AB = BC \ Tan. \ \gamma;$$

if for BC its value be substituted from 2, we obtain

$$AB = \frac{a \ Sin. \ \beta \ Tan. \ \gamma}{Sin. \ (\alpha + \beta)}.$$

Exam. When $\alpha = 79^{\circ}. 45'$, $B = 61^{\circ}. 4'$, $\gamma = 14^{\circ}. 19'. 27''$, $a = 857'$, $AB = 303' \cdot 128$.

Second Solution.

1. If no station can be found which is in the same horizontal plane with B, let CD be any other line in any position whatever; respecting which, for the sake of greater generality, I shall assume, that neither C nor D is in the same horizontal plane with B.

2. Suppose a horizontal plane cBd drawn through B, which cuts the vertical lines Cc, Dd, drawn from C, D, in c, d. It is well known that at C, D the horizontal angles Bcd, Bdc, may be measured, although these points are higher than c, d. Measure the said angle, the station CD, likewise the angle of elevation ACn, and the angle of depression BCn, of the altitude to be measured, Cn being an horizontal line. Let $CD = a$, $Bcd = \alpha$, $Bdc = \beta$, $ACn = \gamma$, $BCn = \delta$.

3. If now we suppose the horizontal line Cm drawn, then also the vertical angle DCm may be measured; let this angle $= \varepsilon$. Now siuce Cm is horizontal, and Dd vertical, consequently DCm is a right-angled triangle, and $\therefore$

$$Cm = a \ Cos. \ \varepsilon.$$

Likewise $cd = Cm$, because $Ccdm$ is a parallelogram.

4. Consequently in the triangle Bcd, the two angles Bcd,

Bdc, and the side *cd* are known; $\therefore$

$$cB = \frac{cd\ Sin.\ \beta}{Sin.\ (\alpha + \beta)} = \frac{a\ Cos.\ \varepsilon\ Sin.\ \beta}{Sin.\ (\alpha + \beta)}.$$

Also $cB = Cn$, because *CcBn* is a parallelogram.

5. Since *AB* is vertical, and *Cn* horizontal; $\therefore$ *ACn*, *BCn*, are right-angled triangles; consequently

$$An = Cn\ .\ Tan.\ \gamma,\ Bn = Cn\ .\ Tan.\ \delta;\ \therefore$$

$$AB = Cn\ (Tan.\ \gamma + Tan.\ \delta) = \frac{Cn\ .\ Sin.\ (\gamma + \delta)}{Cos.\ \gamma\ Cos.\ \delta},$$

or, when we substitute for $Cn = cB$ its value taken from 4,

$$AB = \frac{a\ Cos.\ \varepsilon\ Sin.\ \beta\ Sin.\ (\gamma + \delta)}{Sin.\ (\alpha + \beta)\ Cos.\ \gamma\ Cos.\ \delta}.$$

Cor. By this method, we at the same time find the altitudes of the stations *C*, *D* above the horizontal plane *cBd*; for $Cc = Bn$, $Dd = Dm + md = Dm + Cc$; consequently

$$Cc = \frac{a\ Cos.\ \varepsilon\ Sin.\ \beta\ Tan.\ \delta}{Sin.\ (\alpha + \beta)}$$

$$Dd = \frac{a\ Cos.\ \varepsilon\ Sin.\ \beta\ Tan.\ \delta}{Sin.\ (\alpha + \beta)} + a\ Sin.\ \varepsilon.$$

The measured angles may also be negative; the mode of proceeding in this case is given in the following example.

Exam. Let *AB* represent a church-steeple in a valley, whose height is required to be determined from a measured station *CD* on a neighbouring hill. I assume, that the station *C* is higher than the top of the spire *A*, and *D* lower than *C*, and that the following are known, viz. $a = 357'\cdot 3$, $\alpha = 85^\circ.\ 37'.\ 14''$, $\beta = 79^\circ.\ 13'.\ 12''$, $\gamma = -13^\circ.\ 5'.\ 49''$, $\delta = 20^\circ.\ 18'.\ 9''$, $\varepsilon = -3^\circ.\ 48'.\ 10''$. Now, since whatever angle ϕ may denote, $Cos. -\phi = Cos.\ \phi$, and $Sin. -\phi = -Sin.\ \phi$; we obtain

$$AB = \frac{357'\cdot 3\ Cos.\ 3^\circ.\ 48'.\ 10''\ Sin.\ 79^\circ.\ 13'.\ 12''\ Sin.\ 7^\circ.\ 12'.\ 20''}{Sin.\ 164^\circ.\ 50'.\ 26''\ Cos.\ 13^\circ.\ 5'.\ 49''\ Cos.\ 20^\circ.\ 18'.\ 9''}$$

$$= 183'\cdot 892$$

$$Cc = \frac{357'\cdot 3 \; Cos. 3^{0}.48'.10'' \; Sin. 79^{0}.13'.12'' \; Tan. 20^{0}.18'.9''}{Sin. \; 164^{0}. \; 50'. \; 26}$$
$$= 495'\cdot 470$$
$$Dd = Cc - 357'\cdot 3 \; Sin. \; 3^{0}. \; 48'. \; 10'' = 471'\cdot 773.$$

SECTION LII.

PROB. *A person is standing upon a tower, whose height above the horizontal plane upon which it stands is known: the person wishes, without moving from the spot, to determine the distance between two objects which are in the above-mentioned horizontal plane.*

SOLUT. Let AB (*fig.* 69) represent the tower, whose altitude $= h$; a person standing at A, wishes from it to measure the distance CD, which is in the same horizontal plane CBD with B.

Measure the angles $CAB = \alpha$, $DAB = \beta$, $CAD = \gamma$; then CD may be found. For, since ABC, ABD are two right-angled triangles, and the line AB, together with the angles CAB, DAB are known, therefore AC, AD may be determined: thus

$$AC = h \; sec. \; \alpha, \quad AD = h \; sec. \; \beta.$$

Now, since in the triangle ACD, the sides AC, AD, and the angle CAD, are known, CD may be found: thus

$$CD = h \sqrt{(sec. \; \alpha^2 + sec. \; \beta^2 - 2 \, sec. \; \alpha \; sec. \; \beta \; Cos. \; \gamma)}.$$

EXAM. When $\alpha = 56^{0}. \; 34'$, $\beta = 69^{0}. \; 12'$, $\gamma = 81^{0}. \; 20'$, $h = 214'$, $CD = 665'\cdot 949$.

SECTION LIII.

PROB. *An object standing vertically, for instance a tower, is seen from three stations, which are in a straight line, and in the same horizontal plane upon which the object stands; the distance of these three*

stations from one another, also the angles at which we see the object from each of these stations, are given: find the height of the object, and its distance from each of the three points.

Solut. Let AB (*fig.* 70) be the object, CE any straight line, respecting which it is assumed, that it is in the same horizontal plane with B, and C, D, E are the three stations in this plane. The angles $ACB=\alpha$, $ADB=\beta$, $AEB=\gamma$, likewise the distances $CD=a$, $DE=b$ are given: find the altitude AB, and the distances BC, BD, BE.

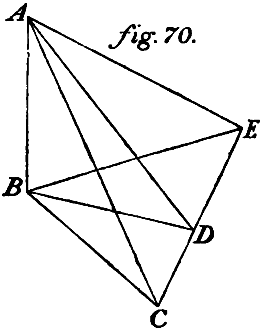


fig. 70.

1. If the line BD can be found, then, from the right-angled triangle ABD, in which both the line BD, and the angle ADB are known, the altitude AB may be calculated. Having found this, then the right-angled triangles ABC, ABE, in which the angles ACB, AEB are known, also give the distances BE, BC. Let $\therefore BD = x$.

2. Then the right-angled triangles ABC, ABD, ABE, give

$$AB = x\,Tan.\,\beta = BC\;Tan.\,\alpha = BE\;Tan.\,\gamma,$$

consequently $BC = \dfrac{x\;Tan.\,\beta}{Tan.\,\alpha}$, $BE = \dfrac{x\;Tan.\,\beta}{Tan.\,\gamma}$

3. But in the triangle BCD,

$$Cos.\,BDC = \frac{BD^2 + CD^2 - BC^2}{2\,BD\,.\,CD} = \frac{x^2 + a^2 - \dfrac{x^2\;Tan^2\,\beta}{Tan.^2\,\alpha}}{2\,ax}$$

and in the triangle BED,

$$Cos.\,BDE = \frac{BD^2 + DE^2 - BE^2}{2\,BD\,.\,DE} = \frac{x^2 + b^2 - \dfrac{x^2\;Tan.^2\,\beta}{Tan.^2\,\gamma}}{2\,bx}.$$

Now since

$BDC+BDE=180^0$: $Cos.\ BDC=-Cos.\ BDE$, $\therefore$

$$\frac{x^2+a^2-\dfrac{x^2\ Tan.^2\ \beta}{Tan.^2\ \alpha}}{2\ ax}=-\frac{x^2+b^2-\dfrac{x^2\ Tan.^2\ \beta}{Tan.^2\ \gamma}}{2\ bx},$$

or $bx^2+a^2b-\dfrac{bx^2\ Tan.^2\ \beta}{Tan.^2\ \alpha}=-ax^2-ab^2+\dfrac{ax^2\ Tan.^2\ \beta}{Tan.^2\ \gamma}$,

or $\dfrac{ax^2\ (Tan.^2\ \beta-Tan.^2\ \gamma)}{Tan.^2\ \gamma}+\dfrac{bx^2\ (Tan.^2\ \beta-Tan.^2\ \alpha)}{Tan.^2\ \alpha}$

$$=(a+b)\ ab.$$

4. Now

$$Tan.^2\ \beta-Tan.^2\ \gamma=(Tan.\ \beta+Tan.\ \gamma)\ (Tan.\ \beta-Tan.\ \gamma)$$
$$=\frac{Sin.\ (\beta+\gamma)\ Sin.\ (\beta-\gamma)}{Cos.^2\ \beta\ Cos.^2\ \gamma}$$

and in like manner

$$Tan.^2\ \beta-Tan.^2\ \alpha=(Tan.\ \beta+Tan.\ \alpha)\ (Tan.\ \beta-Tan.\ \alpha)$$
$$=\frac{Sin.\ (\beta+\alpha)\ Sin.\ (\beta-\alpha)}{Cos.^2\ \beta\ Cos.^2\ \alpha}:$$

also, $Tan.\ \alpha=\dfrac{Sin.\ \alpha}{Cos.\ \alpha}$, $Tan.\ \gamma=\dfrac{Sin.\ \gamma}{Cos.\ \gamma}$. If these values be substituted in the foregoing equation, after the requisite reductions we obtain

$$\frac{ax^2\ Sin.\ (\beta+\gamma)\ Sin.\ (\beta-\gamma)}{Sin.^2\ \gamma\ Cos.^2\ \beta}+\frac{bx^2\ Sin.\ (\beta+\alpha)\ Sin.\ (\beta-\alpha)}{Sin.^2\ \alpha\ Cos.^2\ \beta}$$
$$=(a+b)\ ab$$

and hence

$$x=Sin.\ \alpha\ Sin.\ \gamma\ Cos\ \beta\sqrt{\frac{(a+b)\ ab}{\begin{bmatrix}a\ Sin.(\beta+\gamma)Sin.(\beta-\gamma)Sin.^2\alpha\\+b\ Sin.(\beta+\alpha)Sin.(\beta-\alpha)Sin.^2\gamma\end{bmatrix}}}$$

or when, for the sake of brevity, we substitute

$$a\ Sin.\ (\beta+\gamma)\ Sin.\ (\beta-\gamma)\ Sin.^2\ \alpha=A,$$
$$b\ Sin.\ (\beta+\alpha)\ Sin.\ (\beta-\alpha)\ Sin.^2\ \gamma=B,$$

$$x=Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \beta\sqrt{\frac{(a+b)ab}{A+B}}=BD$$

5. Hence we further obtain

$$AB = x \, Tan. \, \beta = Sin. \, \alpha \, Sin \, \beta \, Sin. \, \gamma \sqrt{\frac{(a+b)\, ab}{A+B}}$$

$$BC = \frac{AB}{Tan. \, \alpha} = Sin. \, \beta \, Sin. \, \gamma \, Cos. \, \alpha \sqrt{\frac{(a+b)\, ab}{A+B}}$$

$$BE = \frac{AB}{Tan. \, \gamma} = Sin. \, \alpha \, Sin. \, \beta \, Cos. \, \gamma \sqrt{\frac{(a+b)\, ab}{A+B}}$$

Exam. Let $\alpha = 19^0. \, 27'. \, 15''$, $\beta = 13^0. \, 4'. \, 7''$, $\gamma = 10^0. \, 48'. \, 25''$, $a = 1750'$, $b = 1047'$. Here $\beta - \alpha = -6^0. \, 23'. \, 8''$, and $Sin. \, (\beta - \alpha) = - Sin. \, 6^0. \, 23'. \, 8''$: consequently B is negative, and we have

$$\begin{aligned} A &= \quad a \, Sin. \, (\beta+\gamma) \, Sin. \, (\beta-\gamma) \, Sin.^2 \alpha = \quad 3{\cdot}100596 \\ B &= - b \, Sin. \, (\beta+\alpha) \, Sin. \, (\alpha-\beta) \, Sin.^2 \gamma = -2{\cdot}200971 \end{aligned}$$

$$A + B = 0{\cdot}899625$$

$$log. \, (A+B) = 0{\cdot}9540615 - 1$$

$$log. \sqrt{\frac{(a+b)\, ab}{A+B}} = 4{\cdot}8778078,$$

hence

$$\begin{aligned} AB &= 1065'{\cdot}754 \\ BC &= 3017'{\cdot}266 \\ BD &= 4591'{\cdot}201 \\ BE &= 5583'{\cdot}320 \end{aligned}$$

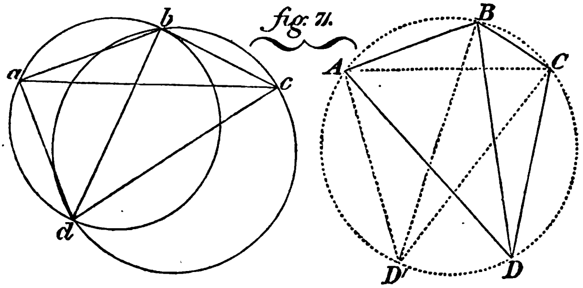
Remark. The visual angles BCA, BDA, BEA, are in fact no other than the angles of elevation of the point A above the horizon, taken from the three stations C, D, E. Now, since it is not necessary to measure towards B, in order to find these angles, we can then, by means of the formulæ already found, determine the altitude of this point A above the horizontal plane, when also from the altitude AB only this point is visible, and in this respect the problem is of great use in practical Geometry.

SECTION LIV.

Prob. *Three places, whose situation is known, are seen from a fourth place, which is in the same plane with*

the others, and there the angle is measured, which the sight-lines make with one another: required to determine the distance of this fourth place from the other three, and also its situation.

Solut. Let A, B, C (*fig.* 71) be the three places; the



distances $AB = a$, $BC = b$, and the angle $ABC = \alpha$ are given. D is the fourth place, where the angles $ADB = \beta$, $BDC = \gamma$ are measured: required to determine the distances AD, BD, CD, and the situation of the point D in reference to A, B, C.

1. If the angle BAD be found; then in the quadrilateral figure $ABCD$ the three angles ABC, BAD, ADC are known, consequently also the fourth BCD. If $\therefore$ in each of the two triangles ABC, DBC, there are two angles and one side known; these two triangles are determined, and the sides AD, BD, CD can be calculated. Let $\therefore$ the unknown angle $BAD = \phi$.

2. Since the four angles of every quadrilateral figure are together $= 4\,R$, $BCD = 360^0 - ABC - ADC - BAD$ $= 360^0 - \alpha - \beta - \gamma - \phi$, or when we abbreviate it by putting $360 - (\alpha + \beta + \gamma) = \mu$, $BCD = \mu - \phi$.

3. From the two angles BAD, ADB, and the side AB of the triangle ABD, we obtain

$$BD = \frac{a\ Sin.\ \phi}{Sin.\ \beta},$$

and from the two angles BCD, BDC, and the side BC of the triangle DBC,

$$BD = \frac{b \, Sin. \, (\mu - \phi)}{Sin. \, \gamma}.$$

We $\therefore$ have

$$\frac{a \, Sin. \, \phi}{Sin. \, \beta} = \frac{b \, Sin. \, (\mu - \phi)}{Sin. \, \gamma},$$

or $\quad a \, Sin. \, \gamma \, Sin. \, \phi = b \, Sin. \, \beta \, Sin. \, (\mu - \phi),$

or also

$a \, Sin. \, \gamma \, Sin. \, \phi = b \, Sin. \, \beta \, (Sin. \, \mu \, Cos. \, \phi - Cos. \, \mu \, Sin. \, \phi)$

If both sides of this equation be divided by $Sin. \, \phi$, we get

$$a \, Sin. \, \gamma = b \, Sin. \, \beta \, (Sin. \, \mu \, Cot. \, \phi - Cos. \, \mu),$$

whence we obtain

$$Cot. \, \phi = Cot. \, \mu + \frac{a \, Sin. \, \gamma}{b \, Sin. \, \beta \, Sin. \, \mu}.$$

4. If the angle ϕ is found; we then have

$$AD = \frac{a \, Sin. \, (\beta + \phi)}{Sin. \, \beta},$$

$$BD = \frac{a \, Sin. \, \phi}{Sin. \, \beta},$$

$$CD = \frac{b \, Sin. \, (\alpha + \beta + \phi - 180^0)}{Sin. \, \gamma}.$$

Cor. 1. Since this problem is of great use in practice, it is $\therefore$ worth while to consider a few particular cases, which are contained in the general solution.

1. If $\alpha = 180^0$, or if the point B falls on the line AC, then we have $\mu = 360^0 - 180^0 - \beta - \gamma = 180^0 - (\beta + \gamma)$; consequently $Cot. \, \mu = - Cot. \, (\beta + \gamma)$, $Sin. \, \mu = Sin. \, (\beta + \gamma)$. We have $\therefore$ for this case

$$Cot. \, \phi = - Cot. \, (\beta + \gamma) + \frac{a \, Sin. \, \gamma}{b \, Sin. \, \beta \, Sin. \, (\beta + \gamma)};$$

or also, because $Cot. (\beta + \gamma) = \frac{Cos. (\beta + \gamma)}{Sin. (\beta + \gamma)}$,

$$Cot. \phi = \frac{a \, Sin. \gamma - b \, Sin. \beta \, Cos. (\beta + \gamma)}{b \, Sin. \beta \, Sin. (\beta + \gamma)}.$$

The first of these expressions is, however, the most convenient for calculation.

2. If the point B be under the line AC, then the convex angle ABC within the quadrilateral figure $ABCD$, and not the concave one ABC, must be taken for α, because the former, and not the latter, together with the three remaining angles of the quadrilateral figure are together $= 4 \, R$, as was supposed in the solution.

3. If $ABC + ADC = 2 \, R$, or $\alpha + \beta + \gamma = 180^\circ$: then $\mu = 180^\circ$, $Cot. \mu = -\infty$, $Sin \, \mu = 0$. We $\therefore$ obtain from 3,

$$Cot. \phi = - \infty + \frac{a \, Sin. \gamma}{0}.$$

Consequently the expression for $Cot. \phi$ appears here in a form, from which its value cannot be determined.

Describe a circle about the triangle ABC; then, because by the hypothesis $ABC + ADC = 2 \, R$, the point D must necessarily fall on the circumference of this circle. This limitation does not obtain, when the angles α, β are assumed to be arbitrary; much more, on account of the given situation of the three points A, B, C, the angles β, γ must be of that magnitude required by the condition, that the point D falls on the circumference of the circle described about the triangle ABC. But under the supposition that the angles β, γ are so assumed that the problem is possible, every point in the circumference $ABCD$ will verify the problem. For let D' be any other in the circumference; then $AD'B = ADB$, $BD'C = BDC$.

As regards the calculation, it is known from Trigonometry, that when ψ denotes any concave or convex angle; for all angles between 90° and 180°, $Cot. \psi = - Cot. (180^\circ - \psi)$; for all angles between 180° and 270°, $Cot. \psi = + Cot. (\psi - 180^\circ)$; and for all angles between 270° and 360°, $Cot. \psi = - Cot. (360^\circ - \psi)$.

Cor. 2. If it is required to find the point D arithmetically only, describe the triangle ABC on paper, then make a triangle abc, which is similar to the former: on ab, as a chord, describe a circular arc adb, which subtends the given angle β; also on bc describe a circular arc bdc, which subtends the given angle γ. The point of intersection d of these two circles, will then give the fourth place on the paper; thus the point d in reference to a, b, c, will have the same situation as the point D has in reference to A, B, C. The reason of this is easily seen.

Exam. 1. Let $a = 1153'{\cdot}7$, $b = 849'{\cdot}43$, $\alpha = 112^0.\ 25'$, $\beta = 27^0.\ 31'$, $\gamma = 19^0.\ 14'$. Here $\mu = 360^0 - (\alpha + \beta + \gamma) = 200^0.\ 50'$; consequently $Sin.\ \mu = -Sin.\ 20^0.\ 50'$, $Cot.\ \mu = Cot.\ 20^0.\ 50'$. We have $\therefore$

$$\frac{a\ Sin.\ \gamma}{b\ Sin.\ \beta\ Sin.\ \mu} = -\frac{1153{\cdot}7\ .\ Sin.\ 19^0.\ 14'}{849{\cdot}43\ Sin.\ 27^0.\ 31'\ Sin.\ 20^0.\ 50'}$$
$$= -2{\cdot}7229400$$
$$Cot.\ \mu = Cot.\ 20^0.\ 50' = 2{\cdot}6279121$$
$$\therefore\ Cot.\ \phi = -0{\cdot}0950279.$$

Since the cotangent here has been found to be negative, ϕ is an obtuse angle. Find $\therefore$ in the Tables an angle to which the positive cotangent $0{\cdot}0950279$ belongs; we find it to be $84^0.\ 34'.\ 18''$. This angle being subtracted from 180^0, gives $\phi = 95^0.\ 25'.\ 42''$; from which the distances AD, BD, CD may be very easily calculated.

Exam. 2. Let $a = 1490'$, $b = 768'$, $\alpha = 235^0$, $\beta = 37^0.\ 10'$. $\gamma = 48^0.\ 15'$. Here $\mu = 39^0.\ 35'$; $\therefore$

$$\frac{a\ Sin.\ \gamma}{b\ Sin.\ \beta\ Sin.\ \mu} = 3{\cdot}7599939$$
$$Cot.\ \mu = 1{\cdot}2095085;$$

consequently $\quad Cot.\ \phi = 4{\cdot}9695024$

and $\quad \phi = 11^0.\ 22'.\ 39''$

In this example it has been assumed, that the point B is on the other side of the line AC, say in B'.

Exam. 3. When $a = 2514'$, $b = 3796'$, $\alpha = 65^0.\ 7'$, $\beta = 135^0.\ 19'$, $\gamma = 113^0.\ 20'$; we find $\phi = 24^0.\ 53'.\ 22''$.

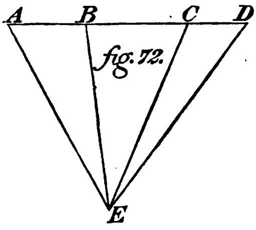
Since in this example the angle $ADC = \beta + \gamma$ is greater than 180^0; consequently the point D cannot be below the line AC, because otherwise $ADC < 180^0$. Nor can it be in the line AC, for then $ADC = 180^0$. Therefore the point D must necessarily be above AC, where, with the points A, C, it forms a convex angle of $135^0.\ 19' + 113^0.\ 20' = 248^0.\ 39'$, or a concave angle of $111^0.\ 21'$; also it must fall within the angle BAC, because the angle $BAD = \phi$ has been found to be positive.

Remark. Of all the Geometricians who have handled this important problem, I can only, for the sake of brevity, adduce the following: Lambert (Mathematical Contributions, Berlin, 1765, p. 73); Tempelhof (Elements of Analytical Finite Magnitudes, Berlin, 1769, p. 482); Langsdorf (Illustration of Kästner's Principles of Analytical Finite Magnitudes, Manheim, 1777, p. 432); Kästner (Geometrical Treatise, 1st Collection, Göttingen, 1793, 2nd part, p. 289); Pfleiderer (Arch. of pure and practical Mathematics, 2nd Number, p. 318).

SECTION LV.

Prob. *Four objects in the same straight line are seen from a station, and there the angle is measured, which the sight-lines make with one another; the distance of the first object from the second, also that of the third from the fourth: find the distance of the second from the third.*

Solut. Let the four objects be A, B, C, D, (*fig.* 72); let E be the station from which they are seen, and the angles $AEB = \alpha$, $BEC = \beta$, $CED = \gamma$, are measured; the distances $AB = a$, $CD = b$ are given: find BC. Let $BC = x$.



1. The triangle AEB gives

$$Sin.\ ABE = \frac{AE\ Sin.\ \alpha}{a},$$

and the triangle BED, in which $BED = \beta + \gamma$, $BD = b + x$

$$Sin.\ DBE = \frac{ED\ Sin.\ (\beta + \gamma)}{b + x}.$$

Now, since $Sin.\ ABE = Sin.\ DBE$,

$$\frac{AE\ Sin.\ \alpha}{a} = \frac{ED\ Sin.\ (\beta + \gamma)}{b + x},$$

and ∴ $$\frac{ED}{AE} = \frac{(b + x)\ Sin.\ \alpha}{a\ Sin.\ (\beta + \gamma)}.$$

2. The triangle ACE, in which $AEC = \alpha + \beta$, $AC = a + x$, gives

$$Sin.\ ACE = \frac{AE\ Sin.\ (\alpha + \beta)}{a + x},$$

and the triangle ECD,

$$Sin.\ DCE = \frac{ED\ Sin.\ \gamma}{b}.$$

Now, since $Sin.\ ACE = Sin.\ DCE$,

$$\frac{AE\ Sin.\ (\alpha + \beta)}{a + x} = \frac{ED\ Sin.\ \gamma}{b},$$

and ∴ $$\frac{ED}{AE} = \frac{b\ Sin.\ (\alpha + \beta)}{(a + x)\ Sin.\ \gamma}.$$

3. If the two expressions found in 1, 2, for $\frac{ED}{AE}$, be put equal to one another, we then obtain

$$(a + x)\ (b + x) = \frac{ab\ Sin.\ (\alpha + \beta)\ Sin.\ (\beta + \gamma)}{Sin.\ \alpha\ Sin.\ \gamma}.$$

The solution of this equation gives

$$x = \sqrt{\left[\left(\frac{a-b}{2}\right)^2 + \frac{ab\ Sin.\ (\alpha + \beta)\ Sin.\ (\beta + \gamma)}{Sin.\ \alpha\ Sin.\ \gamma}\right]} - \frac{a + b}{2}.$$

4. In order to shorten the calculation, give the first part of this expression the following form:

$$\frac{a - b}{2} \sqrt{\left[1 + \frac{4\ ab\ Sin.\ (\alpha + \beta)\ Sin.\ (\beta + \gamma)}{(a - b)^2\ Sin.\ \alpha\ Sin.\ \gamma}\right]}.$$

Then put

$$\frac{2}{a-b}\sqrt{\frac{ab\ Sin.\ (\alpha+\beta)\ Sin.\ (\beta+\gamma)}{Sin.\ \alpha\ Sin.\ \gamma}} = Tan.\ \phi;$$

and find an angle ϕ, such, that its tangent is equal to the left side of the expression: then we have

$$x = \frac{a-b}{2}\sqrt{(1+Tan.^2\phi)} - \frac{a+b}{2}$$

$$= \frac{a-b}{2}\ Sec.\ \phi - \frac{a+b}{2}.$$

Exam. Let $a=2731'$, $b=1987'$, $\alpha=19^0.\ 7'$, $\beta=31^0.\ 5'$, $\gamma=14^0.\ 57'$. Here

$$log.\ \frac{2}{a-b}\sqrt{\frac{ab\ Sin.\ (\alpha+\beta)\ Sin.\ (\beta+\gamma)}{Sin.\ \alpha\ Sin.\ \gamma}} = 1{\cdot}2046761,$$

consequently $\quad log.\ Tan.\ \phi = 11{\cdot}2046761$

$\phi = 86^0.\ 25'.\ 41''{\cdot}63,$

and $\therefore$ $\quad x = 3612'{\cdot}2.$

The angle ϕ must in this case be very accurately calculated, in order to prevent a great mistake. If, for instance, $0{\cdot}63''$ were left out, we should then get for x only $3611'{\cdot}9$.

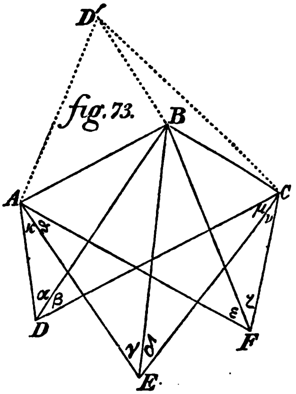
Remark. Another solution of this problem is given by Lambert (Mathematical Contributions, p. 208), which, however, leads to a very difficult formula. My formula agrees essentially with that which Mr. Hauptman Rhode delivered to the Berlin Academy of Science as an Appendix to a Memoir on another subject (Mémoire sur un endroit, &c. Potsdam, chez Horvath, 1804.)

SECTION LVI.

Prob. *The apparent distances of three places from one another, as seen from three different points, are given: likewise the apparent distances of the three points as viewed from one of the three places; required to determine the relative positions of these six points, on the supposition that they are all in the same plane.*

Solut. Let A, B, C, (*fig.* 73) be the three places, which

are seen from the three points D, E, F; the angles $ADB = \alpha$, $BDC = \beta$, $AEB = \gamma$, $BEC = \delta$, $AFB = \varepsilon$, $BFC = \zeta$, also the angles $DAE = \kappa$, $EAF = \lambda$, are given or measured; find the positions of the six points, A, B, C, D, E, F.



1. The angles DCE, ECF, may be determined from the given angles. For since in each triangle the sum of all the angles is equal, by merely inspecting the figure, we have $DCE = \kappa + \alpha + \beta - \gamma - \delta$, $ECF = \lambda + \gamma + \delta - \varepsilon - \zeta$. Since ∴ these angles are known, for shortness' sake put, $DCE = \mu$, $ECF = \nu$. If, besides, the angles EAB, ECB, are known, we then have all the angles of the figure, and consequently also the positions of the six points. Let, therefore, $EAB = \phi$, $ECB = \psi$.

2. Since $DAB = \phi + \kappa$, $DCB = \psi - \mu$; consequently in the triangle DAB we have

$$BD = \frac{AB \ Sin. \ (\phi + \kappa)}{Sin. \ \alpha}$$

and in the triangle BCD

$$BD = \frac{BC \ Sin. \ (\psi - \mu)}{Sin. \ \beta}.$$

These two expressions for BD, when put equal to one another, give

$$\frac{AB}{BC} = \frac{Sin. \ \alpha \ Sin. \ (\psi - \mu)}{Sin. \ \beta \ Sin. \ (\phi + \kappa)}.$$

3. In like manner, from the two triangles EAB, ECB, we obtain

$$BE = \frac{AB \ Sin. \ \phi}{Sin. \ \gamma}, \quad BE = \frac{BC \ Sin. \ \psi}{Sin. \ \delta},$$

and ∴

$$\frac{AB}{BC} = \frac{Sin. \ \gamma \ Sin. \ \psi}{Sin. \ \delta \ Sin. \ \phi}.$$

4. Further, from the two triangles FAB, FCB, in which $FAB = \phi - \lambda$, $FCB = \psi + \nu$, we get

$$BF = \frac{AB \; Sin. \; (\phi - \lambda)}{Sin. \; \varepsilon}, \; BF = \frac{BC \; Sin. \; (\psi + \nu)}{Sin. \; \zeta},$$

and $\therefore$

$$\frac{AB}{BC} = \frac{Sin. \; \varepsilon \; Sin. \; (\psi + \nu)}{Sin. \; \zeta \; Sin. \; (\phi - \lambda)}.$$

5. If we put the expressions found for $\frac{AB}{BC}$, in 2, 3, 4, equal to one another, we then obtain

$$\frac{Sin. \; \gamma \; Sin. \; \psi}{Sin. \; \delta \; Sin. \; \phi} = \frac{Sin. \; \alpha \; Sin. \; (\psi - \mu)}{Sin. \; \beta \; Sin. \; (\phi - \kappa)}$$

$$\frac{Sin. \; \gamma \; Sin. \; \psi}{Sin. \; \delta \; Sin. \; \phi} = \frac{Sin. \; \varepsilon \; Sin. \; (\psi + \nu)}{Sin. \; \zeta \; Sin. \; (\phi - \lambda)}.$$

If we expand $Sin. \; (\psi - \mu)$, $Sin. \; (\phi + \kappa)$, $Sin. \; (\psi + \nu)$, $Sin. \; (\phi - \lambda)$ and multiply cross ways, these equations give

$$Sin. \; \beta \; Sin. \; \gamma \; Sin. \; \psi \; (Sin. \; \phi \; Cos \; \kappa + Cos. \; \phi \; Sin. \; \kappa)$$
$$= Sin. \; \alpha \; Sin. \; \delta \; Sin. \; \phi \; (Sin. \; \psi \; Cos. \; \mu - Cos. \; \psi \; Sin. \; \mu)$$

$$Sin. \; \gamma \; Sin. \; \zeta \; Sin. \; \psi \; (Sin. \; \phi \; Cos. \; \lambda - Cos. \; \phi \; Sin. \; \lambda)$$
$$= Sin. \; \delta \; Sin. \; \varepsilon \; Sin. \; \phi \; (Sin. \; \psi \; Cos. \; \nu + Cos. \; \psi \; Sin. \; \nu)$$

If each of these equations be divided by $Sin. \; \phi \; Sin. \; \psi$, we have

$$Sin. \; \beta \; Sin. \; \gamma \; (Cos. \; \kappa + Cot. \; \phi \; Sin. \; \kappa)$$
$$= Sin. \; \alpha \; Sin. \; \delta \; (Cos. \; \mu - Cot. \; \psi \; Sin. \; \mu)$$

$$Sin. \; \gamma \; Sin. \; \zeta \; (Cos. \; \lambda - Cot. \; \phi \; Sin. \; \lambda)$$
$$= Sin. \; \delta \; Sin. \; \varepsilon \; (Cos. \; \nu + Cot. \; \psi \; Sin. \; \nu).$$

6. The first of the two equations last found gives

$$Cot. \; \psi = \frac{Sin. \; \alpha \; Sin. \; \delta \; Cos. \; \mu - Sin. \; \beta \; Sin. \; \gamma \; (Cos. \; \kappa + Cot. \; \phi \; Sin. \; \kappa)}{Sin. \; \alpha \; Sin. \; \delta \; Sin. \; \mu}$$

and the second

$$Cot. \; \psi = \frac{Sin. \; \gamma \; Sin. \; \zeta \; (Cos. \; \lambda - Cot. \; \phi \; Sin. \; \lambda) - Sin. \; \delta \; Sin. \; \varepsilon \; Cos. \; \nu}{Sin. \; \delta \; Sin. \; \varepsilon \; Sin. \; \nu}$$

If these two expressions be put equal to one another, we obtain

$$Cot.\ \phi = \frac{\left[\begin{array}{l} Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \varepsilon\ (Sin.\ \nu\ Cos.\ \mu + Sin.\ \mu\ Cos.\ \nu) - \\ Sin.\beta\ Sin.\gamma\ Sin.\varepsilon\ Cos.\nu\ Cos.\kappa - Sin.\alpha\ Sin.\gamma\ Sin\ \zeta\ Sin.\mu\ Cos.\lambda \end{array}\right]}{Sin\ \beta\ Sin.\gamma\ Sin.\varepsilon\ Sin.\nu\ Sin.\kappa - Sin.\alpha\ Sin.\gamma\ Sin.\zeta\ Sin.\lambda\ Sin.\mu}$$

or, if the numerator and denominator be divided by $Sin.\ \gamma$, and $Sin.\ (\mu + \nu)$ be substituted for $Sin.\ \nu\ Cos.\ \mu + Sin.\ \mu\ Cos.\ \nu$, and $Cosec.\ \gamma$ for $\frac{1}{Sin.\ \gamma}$

$$Cot.\ \phi = \frac{\left[\begin{array}{l} Cosec.\ \gamma\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \varepsilon\ Sin.\ (\mu + \nu) - \\ Sin.\ \beta\ Sin.\ \varepsilon\ Sin.\ \nu\ Cos.\ \kappa - Sin.\ \alpha\ Sin.\ \zeta\ Sin.\ \mu\ Cos.\ \lambda \end{array}\right]}{Sin.\ \beta\ Sin.\ \varepsilon\ Sin.\ \nu\ Sin.\ \kappa - Sin.\ \alpha\ Sin.\ \zeta\ Sin.\ \lambda\ Sin.\ \mu}$$

7. In order to determine $Cot.\ \psi$, it is only requisite to eliminate $Cot.\ \phi$ from the two equations last found in 5; but this object will be much more easily attained by substituting in the expression found for $Cot.\ \phi$ the angles

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \kappa,\ \lambda,\ \mu,\ \nu,$$

respectively for the angles

$$\zeta,\ \varepsilon,\ \delta,\ \gamma,\ \beta,\ \alpha,\ \nu,\ \mu,\ \lambda,\ \kappa;$$

because the former have the same position with respect to the angle ψ, that the last have with respect to ϕ. By this substitution, we obtain

$$Cot.\ \psi = \frac{\left[\begin{array}{l} Cosec.\ \delta\ Sin.\ \zeta\ Sin.\ \gamma\ Sin.\ \beta\ Sin.\ (\lambda + \kappa) - \\ Sin.\ \varepsilon\ Sin.\ \beta\ Sin.\ \kappa\ Cos.\ \nu - Sin.\ \zeta\ Sin.\ \alpha\ Sin.\ \lambda\ Cos.\ \mu \end{array}\right]}{Sin.\ \varepsilon\ Sin.\ \beta\ Sin.\ \kappa\ Sin.\ \nu - Sin.\ \zeta\ Sin.\ \alpha\ Sin.\ \mu\ Sin.\ \lambda}$$

in which expression the denominator is the same as that in the expression for $Cot.\ \phi$.

In these two expressions for $Cot.\ \phi$, $Cot.\ \psi$,

$$\mu = \kappa + \alpha + \beta - \gamma - \delta,\quad \nu = \lambda + \gamma + \delta - \varepsilon - \zeta;$$

consequently $\quad \mu + \nu = \kappa + \lambda + \alpha + \beta - \varepsilon - \zeta.$

EXAM. Let $\alpha = 40^{\circ}.\ 36'$, $\beta = 27^{\circ}.\ 9'$, $\gamma = 36^{\circ}.\ 49'$, $\delta = 31^{\circ}.\ 18'$, $\varepsilon = 22^{\circ}$, $\zeta = 13^{\circ}.\ 28'$, $\kappa = 25^{\circ}.\ 23'$, $\lambda = 20^{\circ}.\ 17'$; $\therefore\ \mu = 25^{\circ}.\ 1'$, $\nu = 52^{\circ}.\ 56'$. Here

$$\begin{aligned}
&Cosec.\ \gamma\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \varepsilon\ Sin.\ (\mu + \nu) &= 0{\cdot}20668928\\
&Sin.\ \beta\ Sin.\ \varepsilon\ Sin.\ \nu\ Cos.\ \kappa &= 0{\cdot}12323178\\
&Sin.\ \alpha\ Sin.\ \zeta\ Sin\ \ \mu\ Cos.\ \lambda &= 0\ 06011447\\
&Cosec.\ \delta\ Sin.\ \zeta\ Sin.\ \gamma\ Sin.\ \beta\ Sin.\ (\lambda + \kappa) &= 0{\cdot}08767862\\
&Sin.\ \varepsilon\ Sin.\ \beta\ Sin.\ \kappa\ Cos.\ \nu &= 0{\cdot}04416768\\
&Sin.\ \zeta\ Sin.\ \alpha\ Sin.\ \lambda\ Cos.\ \mu &= 0{\cdot}04760865\\
&Sin.\ \beta\ Sin.\ \varepsilon\ Sin.\ \nu\ Sin.\ \kappa &= 0{\cdot}05847082\\
&Sin.\ \alpha\ Sin.\ \zeta\ Sin.\ \lambda\ Sin.\ \mu &= 0{\cdot}02221714
\end{aligned}$$

Therefore,

$$Cot.\ \phi = \frac{0{\cdot}02334303}{0{\cdot}03625368} = 0{\cdot}6438802$$

$$Cot.\ \psi = \frac{-0{\cdot}00409771}{0{\cdot}03625368} = -\ 0{\cdot}1130288$$

consequently $\phi = 57^{\circ}.\ 13'.\ 24''$, $\psi = 96^{\circ}.\ 26'.\ 55''$.

When the angles ϕ and ψ are found, it is easy to determine the points A, B, C, D, E, F, and also to calculate the distances of all these points from one another, when only one of these distances (no matter which) has either been measured directly, or is otherwise already known.

COR. When the stations D, E, F, have a different position from that assumed in *fig.* 73, then also the formulæ found for $Cot.\ \phi$, $Cot.\ \psi$ will still obtain, provided the values corresponding to this position are given to the angles α, β, γ, δ, &c. The mode of proceeding in this case will be best elucidated by an example.

Suppose the first station is at D' instead of D, and that it is wished to determine the values of α, β, κ depending upon the position of this point; then suppose that the point D has arrived by slow degrees at D'. Now, as the point D advances to D', and approaches the line AB, the angle $ADB = \alpha$ will always increase, till this point falls in AB itself, and $\alpha = 180^{\circ}$. If the point D move towards the upper side of the line AB, then the concave angle will become a convex one, and consequently, if we assume the point D at D', we must under-

stand by α, not the concave, but the convex angle $AD'B$. But if, as is desirable in most cases, we wish to avoid the convex angle, we can then let the point D pass over to D', by producing AB, say towards the side A. Under this supposition, the angle α will constantly decrease, because the lines DB, DA approach each other, till it becomes $= 0$, when the point D is situated in that part of BA which is produced, and the lines DB, DA coincide; and lastly, negative, when the point D moves towards the upper side of AB, and the line BD, which before was on this side of the line DA, is now on the other side of it. If $\therefore$ we make use of the concave angle $AD'B$, we must put $\alpha = -AD'B$.

Proceed in like manner with the angle β. Thus, while the point D is situated under the line BC, or BC produced, and consequently the line DC is on this side of the line DB, the point D is positive; but when it is above the line BC, and $\therefore$ the line DC is on the other side of DB, it is negative. Thus, for the point D', $\beta = -BD'C$.

The angle κ is positive, while it is under AE, but negative as soon as the line AD is on the other side of the line AE. Thus, for the point D', $\kappa = -EAD'$.

With regard to the trigonometrical functions of the negative angles, when θ denotes any angle, $Sin. -\theta = -Sin.\ \theta$, $Cos. -\theta = Cos.\ \theta$, $Tan. -\theta = -Tan.\ \theta$, $Cot. -\theta = -Cot.\ \theta$, $Sec. -\theta = Sec.\ \theta$, $Cosec. -\theta = -Cosec.\ \theta$. Calculations involving angles of this kind occur frequently in the sequel.

REMARK. Lambert's problem which is here solved, is of great practical use; because by its means the positions of six points are obtained at once; it is to be found in his Contributions, I. p. 72. The formula there given (p. 82) for *Cot.* x, or mine for *Cot.* φ, is not quite correct, because both in the numerator and denominator a factor has been omitted by mistake. This error was first discovered by Good (Lambert's Scientific Correspondence, 2nd vol. p. 232), and acknowledged (*ibid.* p. 236) by Lambert.

VI. PROBLEMS ON THE CIRCLE.

SECTION LVII.

Prob. *The radius of a circle being given, to calculate from it the circumference and area of the circle.*

Solut. Let the radius of a circle $= r$, the circumference $= p$, and the area $= q$; then, as is already known,

$$p = 2\pi r, \; q = \pi r^2;$$

or, when the diameter $= d$,

$$p = \pi d, \; q = \tfrac{1}{4}\pi d^2:$$

in which π is the number which represents the circumference of a circle, whose diameter $= 1$. This number is

$$3{\cdot}14159265358979323846264338327950\ldots$$

Thus far has this number been calculated by Ludolph of Cologne. In Vega's large and small Logarithmic Tables, this number is calculated by the author as far as 143 decimal places. In most practical cases, however, it will be necessary merely to make use of the first five decimal places; and thus $\pi = 3{\cdot}14159$.

By the transformation of this number into a continued fraction, we obtain the following abbreviated values:

$$\frac{3}{1}, \; \frac{22}{7}, \; \frac{333}{106}, \; \frac{355}{113}, \; \frac{103993}{33102}, \; \frac{104348}{33215}, \; \frac{208341}{66317}, \; \&c.$$

Lambert, in his Correspondence, II. pp. 156, 157, calculates 27 values of this kind; the two last, however, are not correct, as Professor Schulz has discovered. (Solution of some of the most important Mathematical Theories, Königsberg, 1803, p. 159.)

In practice, when great accuracy is not requisite, we may put $\pi = \frac{355}{113}$, because this number differs from Ludolph's, beginning from the seventh decimal place: for $\frac{355}{113} =$ 3·1415929 ...

If we make use of logarithms; then

$$log.\ \pi = 0{\cdot}4971498726941338543512 7 \ldots$$

or abbreviated

$$log.\ \pi = 0{\cdot}4971499.$$

Cor. From these two equations, for p and q we obtain

$$r = \frac{p}{2\pi},\ r = \sqrt{\frac{q}{\pi}},\ q = \frac{p^2}{4\pi},\ p = 2\sqrt{\pi q}.$$

Exam. 1. The diameter of a circle is $42'.\ 1''.\ 2'''$: what are its circumference and area? Ans. The circumference $= 132'.\ 3''.\ 2'''$, and the area $= 1393\ \square'.\ 37\ \square''.\ 04\ \square'''$, nearly.

Exam. 2. The area of a circle is $3765\ \square'.\ 18\ \square''$: what are the radius and circumference? Ans. The radius $= 33'.\ 6''.\ 2'''$, and the circumference $= 217'.\ 5''.\ 2'''$, nearly.

A few Examples in this Rule.

I. There are three circles given; the diameter of the first is $9'.\ 7''$, the diameter of the second is $13'.\ 6''$, and the diameter of the third is $22'.\ 9''$: find a circle, whose area is equal to the sum of the areas of these three circles. What is the diameter of this circle? Ans. About $28'.\ 3''.\ 4'''$.

II. From any point let there be two concentric circles described; let the radius of the exterior circle $= 1'.\ 5''.\ 3'''$, and the radius of the interior one $= 10''.\ 9'''$: what is the radius of a circle, whose area is equal to the area of the circle described between these two? Ans. About $10''.\ 7'''.\ 4^{\text{iv}}$.

III. The radius of a circle is $39'.\ 8''$: it is required to

describe about it another concentric circle, such that the area of the circle described between these two = 385 □′. What is its area? Ans. About 41′. 3″. 1‴.

IV. The diameter of a circle is 45′. 3″. 7‴: find the diameter of another circle, whose area : the area of the former circle : : 387 : 932. What is the diameter?
Ans. 29′. 2″. 3‴, nearly.

V. There are two circles given; the circumference of one is 69′. 5″, and that of the other 35′. 9″: what is the diameter of a circle, whose area is equal to the sum of the areas of these two circles? Ans. 24′. 8″. 9‴, nearly.

VI. To convert a circle, whose diameter is 9′. 7″, into an equilateral triangle: what is the dimension of a side of this triangle? Ans. 13′. 0″. 6‴ nearly.

SECTION LVIII.

Prob. *From the given radius of a circle, to find the value of an arc expressed in degrees, minutes, and seconds.*

Solut. Let the radius of a circle $= r$, the given number of degrees of an arc $= \phi$; let the length of this arc, expressed in the same terms of unity as the radius, $= l$. Since the circumference of the whole described circle, with the radius r, $= 2\pi r$, then

$$360^0 : \phi = 2\pi r : l,$$

and ∴

$$l = \frac{\pi r \phi}{180^0}.$$

If ϕ, besides degrees, contains also minutes and seconds; the degrees must be converted into minutes or seconds, and the 180 in the denominator multiplied by 60, or by $60 \times 60 = 3600$. If it is considered preferable, the minutes and seconds can be converted into decimal parts of a degree, and the 180 remain unaltered.

Cor. Conversely, from the expression found for l, we obtain

$$\phi = \frac{180^\circ l}{\pi r},\ r = \frac{180^\circ l}{\pi \phi}.$$

Exam. 1. What is the length of an arc of $37^\circ.\ 19'$, in a circle whose radius is $13'.\ 4''$?

Ans. $87'.\ 2''.\ 7'''.\ 4^{iv}$, nearly.

Exam. 2. What is the length of an arc of $149^\circ.\ 16'.\ 13''$, in a circle whose radius is $19'.\ 7''$?

Ans. $51'.\ 3''.\ 2'''.\ 3^{iv}$, nearly.

Exam. 3. What is the length of an arc of $253^\circ.\ 9'.\ 3''$, in a circle, whose radius is $23'.\ 8''.\ 6'''$?

Ans. $105'.\ 4''.\ 2'''.\ 1^{iv}$, nearly.

Exam. 4. How many degrees, minutes, and seconds, does an arc contain, whose chord is $25'.\ 7''$, when the radius of the circle to which it belongs $= 19'.\ 3''.\ 7'''$?

Ans. $76^\circ.\ 1'.\ 11''$, nearly.

Exam. 5. What is the radius of a circle, when an arc of $25^\circ.\ 3'.\ 49''$ has a chord of $247'.\ 8''$?

Ans. $566'.\ 4''.\ 7'''.\ 4^{iv}$, nearly.

Exam. 6. There are two arcs having equal chords, which belong to two different circles; one is $15^\circ.\ 39'.\ 7''$, the other $56^\circ.\ 9'.\ 43''$; the first belongs to a circle whose radius is $7'.\ 6''.\ 3'''$: what is the radius of the circle to which the other arc belongs? Ans. $2'.\ 1''.\ 2'''.\ 6^{iv}$ nearly.

SECTION LIX.

Prob. *From the given angle and radius of a segment of a circle, to find its area.*

Solut. Let ϕ be the number of degrees, minutes and seconds, which the angle, and consequently also the arc, of the circle contains, l the chord of the arc, r the radius, and

q the area of the segment; then, by the foregoing §, $l = \frac{\pi r \phi}{180^\circ}$.

Now, since every segment of a circle is equal to a triangle, whose base is the arc, and whose altitude is the radius of this segment, consequently

$$q = \frac{lr}{2} = \frac{\pi r^2 \phi}{360^\circ}.$$

Cor. Hence we obtain

$$\phi = \frac{360^\circ q}{\pi r^2}, \quad r = \sqrt{\frac{360^\circ q}{\pi \phi}}.$$

Exam. 1. The radius of a circle is $7''.\ 9'''$; the angle of a segment of this circle contains $37^\circ.\ 5'$: what is the area of this segment? Ans. $20 \,\square''.\ 19 \,\square'''.\ 67 \,\square^{IV}$, nearly.

Exam. 2. What is the angle of a segment, whose radius $= 25783'$, and whose area $= 935 \,\square'$?

Ans. $0{\cdot}58023''$, nearly.

Exam. What is the radius of a segment, whose angle $= 46^\circ.\ 25'.\ 18''$, and whose area $= 367 \,\square'.\ 90 \,\square''$?

Ans. $301'.\ 3''.\ 5'''$, nearly.

A few practical Examples to this Rule.

I. The area of a segment is equal to the square of its radius: what is its angle? Ans. $114^\circ.\ 35'.\ 29''$.

II. There is a segment, whose angle $= 69^\circ.\ 47''$, and is such, that when the radius, arc, and area, in the order in which they are here placed, are expressed in terms of one and the same unity for the chord and area, the three values thus obtained are in continual proportion: what is the area of this segment, the foot calculated at unity?

Ans. $13008 \,\square'.\ 28 \,\square''$.

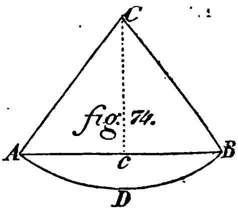
III. Find a triangle, whose three sides are in the proportion of the three numbers 11, 13, 20, and whose area is equal

to the area of a segment, whose angle is 19°. 27′. 5″, and whose arc is 27′. 3″. What are the three sides of this triangle? Ans. One side = 44′·8594, another = 53·0157, and the third = 81′·5626.

SECTION LX.

Prob. *To find the area of a segment of a circle, from the arc, and the radius of this arc, expressed in degrees, minutes and seconds.*

Solut. Let *ADB* (*fig.* 74) = ϕ be the given arc expressed in degrees, minutes, and seconds; *C* the center, and *r* the radius of the arc; then, by the foregoing §,



$$\text{Segment } ACB = \frac{\pi r^2 \phi}{360^0}.$$

Now $\triangle ACB = \frac{1}{2} r^2 Sin. \phi$, (§ XXVI)

$$\text{consequently Segment } ADBA = \frac{\pi r^2 \phi}{360^0} - \tfrac{1}{2} r^2 Sin. \phi$$

$$= \tfrac{1}{2} r^2 \left[\frac{\pi \phi}{180^0} - Sin. \phi\right].$$

Exam. 1. When $\phi = 29^0.\ 38'.\ 15''$, $r = 37'$, then segment $ADBA = 15{\cdot}5801\ \square'$.

Exam. 2. When $\phi = 73^0.\ 25'.\ 11''$, $r = 65'$, then segment $ADBA = 682{\cdot}3271\ \square'$.

SECTION LXI.

Prob. *From the given chord and radius of a segment, to find its area.*

Solut. Let *ACB* (*fig.* 74) be the segment, whose area is sought, let the given radius = *r*, the given chord *AB* = *a*.

If the angle ACB is found, then the area of the segment is also found (§ LIX).

Put $ACB = \phi$: then, if the perpendicular Cc is drawn, $ACc = \frac{1}{2}\phi$, $Ac = r\,Sin.\,\frac{1}{2}\phi$, $AB = a = 2\,r\,Sin.\,\frac{1}{2}\phi$; consequently

$$Sin.\,\tfrac{1}{2}\phi = \frac{a}{2\,r},$$

from which the angle ϕ may be determined. Having found this angle, then

$$\text{Segment } ACB = \frac{\pi r^2 \phi}{360^0}.$$

Cor. If the chord AB and the angle ACB be given, then in like manner we obtain the area of the segment from § LIX, by substituting in the expression found in this section for r its value $\frac{a}{2\,Sin.\,\frac{1}{2}\phi}$. Thus

$$\text{Segment } ACB = \frac{\pi a^2 \phi}{1440^0\,Sin.^2\,\frac{1}{2}\phi}:$$

which expression is most readily calculated by means of logarithms.

Exam. 1. The chord of a segment $= 23'$, the radius $= 29'.\,7''$; what is its area? Ans. $350\,\square'.\,71\,\square''$.

Exam. 2. The chord of a segment $= 54'$, the angle $= 67^0.\,15'.\,25''$: what is its area? Ans. $1395\,\square'.\,07\,\square''$.

Remark. If the chord be equal to the radius, then the area of the segment $= \frac{1}{6}\pi r^2$, because in this case the chord is a side of a regular hexagon described in the circle of which the segment is a part. If the chord $= r\sqrt{2}$, then the segment is a quadrant; and consequently its area $= \frac{1}{4}\pi r^2$. If the chord $= r\sqrt{3}$, then it is a side of a triangle described in the circle; and ∴ the segment $= \frac{1}{3}\pi r^2$.

SECTION LXII.

Prob. *From the arc and chord of a segment expressed in degrees, minutes, and seconds, to find the area which is included by the chord and arc.*

Solut. By the foregoing section, if we retain the notation

there used (*fig.* 74),

$$\text{Segment } ACB = \frac{\pi\, a^2\, \phi}{1440^0\ Sin.^2\ \frac{1}{2}\,\phi}.$$

But in the triangle ACB, $Cc = \frac{1}{2}\, a\ Cot.\ \frac{1}{2}\,\phi$, and consequently

$$\triangle\ ACB = \tfrac{1}{4}\, a^2\ Cot.\ \tfrac{1}{2}\,\phi\,;$$

we ∴ have

$$\text{Segment } ABDA = \frac{\pi\, a^2\, \phi}{1440^0\ Sin.^2\ \frac{1}{2}\,\phi} - \tfrac{1}{4}\, a^2\ Cot.\ \tfrac{1}{2}\,\phi.$$

EXAM. 1. When $a = 18'.\ 9''.\ 3'''$, $\phi = 113^0.\ 39'.\ 25''$, Segment $ABDA = 68{\cdot}2624\ \square'$.

EXAM. 2. When $a = 126'.\ 5''.\ 8'''$, $\phi = 269^0.\ 14'.\ 7''$, Segment $ABDA = 22527{\cdot}207\ \square'$.

REMARK. In the second example, $Cot.\ \frac{1}{2}\,\phi = Cot.\ 134^\circ.\ 37'.\ 3''{\cdot}5 = -\ Cot.\ 45^\circ.\ 22'.\ 56''{\cdot}5$; the section is greater than the segment, because it is greater than a semicircle.

SECTION LXIII.

PROB. *To find a segment of a circle, which is bisected by its chord.*

SOLUT. Let ACB (*fig.* 74) be the required segment, which is bisected by its chord AB, so that the segment $ADBA = \triangle\ ACB$, or $2\ \triangle\ ACB =$ segment ACB. Now the segment $ACB = \frac{\pi\, r^2\, \phi}{360^0}$ (§ LIX), and $\triangle\ ACB = \frac{1}{2}\, r^2\ Sin.\ \phi$; we therefore have the equation,

$$\frac{\pi\, r^2\, \phi}{360^0} = r^2\ Sin.\ \phi,$$

or

$$\frac{\pi\, \phi}{360^0} = Sin.\ \phi.$$

2. In order to solve an equation of this kind between an arc and its sine, there is scarcely a more convenient method, than that which the rule known by the name of False Position

presents. To apply this mode of calculation with advantage, make first a few rough guesses, in order at least to approximate somewhat near to the value of the angle ϕ. Assume $\phi = 90^0$, this gives, $\frac{\pi\,\phi}{360^0} = \frac{1}{4}\,\pi = 0{\cdot}785 \ldots$ and $Sin.\,\phi = 1$; consequently ϕ must be $> 90^0$. If we assume $\phi = 120^0$; then $\frac{\pi\,\phi}{360} = \frac{1}{3}\,\pi = 1{\cdot}047 \ldots$ and $Sin.\,\phi = 0{\cdot}866 \ldots$; consequently $\phi < 120^0$. Hence it follows, that ϕ must be between 90 and 120^0. Put ∴ successively $\phi = 100^0$, $\phi = 110^0$; then we have,

When $\phi = 100^0$.		When $\phi = 110^0$.	
$log.\,\frac{\pi\phi}{360^0} =$	$0{\cdot}9408474 - 1$	$log.\,\frac{\pi\phi}{360^0} =$	$0{\cdot}9822401 - 1$
$log.\,Sin.\,\phi =$	$9{\cdot}9933515 - 10$	$log.\,Sin.\,\phi =$	$9{\cdot}9729858 - 10$
Error	525041	Error	-92543
Subtract	-92543		

617584 Difference of errors.

Now form the following proportion:

$$617584 : 525041 = 10^0 : 8^0\ 30' \text{ nearly.}$$

We have ∴ $\phi = 108^0.\ 30'$, nearly.

3. This value found for ϕ does not differ very much from the real one, as the following calculation shews. In order to render it more accurate, try also further the assumption $\phi = 108^0.\ 35'$: we then have

when $\phi = 108^0.\ 30'$.		When $\phi = 108^0.\ 35'$.	
$log.\,\frac{\pi\phi}{360^0} =$	$0{\cdot}9762771 - 1$	$log.\,\frac{\pi\phi}{360^0} =$	$0{\cdot}9766105 - 1$
$log.\,Sin.\,\phi =$	$9{\cdot}9769566 - 10$	$log.\,Sin.\,\phi =$	$9{\cdot}9767447 - 10$
Error	6795	Error	·1342
Subtract	1342		

5453 Differencc of errors.

Now form the following proportion:

$$5453 : 1342 = 5' : 1'.\ 13'' \text{ nearly.}$$

We $\therefore$ have $\phi = 108^0.\ 35' + 1'\ 13'' = 108^0.\ 36'.\ 13''$, nearly.

4. Since the value of ϕ is only a little too small, as the following calculation shews, $\therefore$ assume ϕ only a few seconds larger; we then have

When $\phi = 108^0.\ 36'.\ 13''$.		When $\phi = 108^0.\ 36'.\ 14''$.	
$log.\ \dfrac{\pi\phi}{360^0}$	$= 0{\cdot}9766916 - 1$	$log.\ \dfrac{\pi\phi}{360^0}$	$= 0{\cdot}9766927 - 1$
$log.\ Sin.\ \phi$	$= 9{\cdot}9766930 - 10$	$log.\ Sin.\ \phi$	$= 9{\cdot}9766923 - 10$
Error	14	Error	-4
Subtract	-4		
	18 Difference of errors.		

Put $\therefore$ again:

$$18 : 4 = 1'' : 13'''.$$

5. Since this last approximation is only represented by $'''$, this denotes that even the seconds in the values of ϕ found in 4 are correct, and we $\therefore$ can put $\phi = 108^0.\ 36'.\ 13''$. If we wish to determine this value still more accurately, we must use logarithms with more than seven decimal places. Euler (Introduction to the Analysis Infinitorum, translated by Michelsen, 2nd vol. p. 452) performed this; and by these means found

$$\phi = 108^0.\ 36'.\ 13''.\ 45'''.\ 27^{IV}.\ 6^{V}.$$

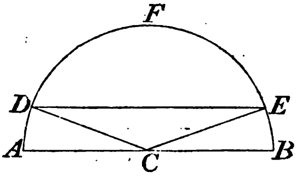
REMARK. The method here adopted is founded on the well-known property of trigonometrical lines and their logarithms, that their differences, in small alterations of the angles, are nearly as the differences of the angles themselves, which supposition is the more correct, the smaller these changes are with respect to the angle.

SECTION LXIV.

PROB. *To bisect a semicircle by a chord which is parallel to the diameter.*

SOLUT. Let $ADFEB$ (*fig.* 75) be a semicircle, and DE $\parallel$ the chord AB, which bisects it; let the radius $= r$, and the angle $DCE = \phi$.

fig. 75.



1. Since the area of the semicircle $= \frac{1}{2}\pi r^2$, and the area of the segment $DFED = \frac{\pi r^2 \phi}{360^0} - \frac{1}{2} r^2 \, Sin. \, \phi$ (§ LX), we have the equation

$$\tfrac{1}{4}\pi r^2 = \frac{\pi r^2 \phi}{360^0} - \tfrac{1}{2} r^2 \, Sin. \, \phi,$$

or
$$90^0 \pi = \pi \phi - 180^0 \, Sin. \, \phi,$$

or also
$$\frac{\pi(\phi - 90^0)}{180^0} = Sin. \, \phi.$$

Since ϕ must necessarily be greater than 90^0, put $\phi = 90^0 + \psi$; this gives $\phi = Cos. \, \psi$, and instead of the foregoing equation, we then have the one,

$$\frac{\pi \psi}{180^0} = Cos. \, \psi$$

2. Hence the value of ψ may be determined in the same way as in the foregoing §. Since ψ must be $< 90^0$, because $\phi < 180$; try $\psi = 45^0$. This assumption gives, $\frac{\pi\psi}{180^0} = \frac{1}{4}\pi = 0{\cdot}785 \ldots$, $Cos. \, \psi = 0{\cdot}707 \ldots$; whence we conclude, that ψ is actually less than 45^0, but cannot be widely different from it, because these two magnitudes first differ in the second decimal place. If $\therefore$ we put $\psi = 40^0$; then we have $\frac{\pi\psi}{180^0} = \frac{2}{9}\pi = 0{\cdot}698 \ldots$, and $Cos. \, \psi = 0{\cdot}766 \ldots$,

and consequently $\psi > 40^0$. After having in this way previously convinced ourselves, that the value of ψ lies between 40^0 and 45^0; we can then approximate it more nearly by means of the proportional parts. To effect this, try the two assumptions $\psi = 41^0$, $\psi = 43^0$.

$\psi = 41^0$.	$\psi = 43^0$.
$log. \frac{\pi\psi}{180^0} = 0{\cdot}8546613 - 1$	$log. \frac{\pi\psi}{180^0} = 0{\cdot}8753459 - 1$
$log. Cos. \psi = 9{\cdot}8777799 - 10$	$log. Cos. \psi = 9{\cdot}8641275 - 10$
Error 231186	Error $-$ 112184
Subtract $-$ 112184	

343370 Difference of errors.

Now put

$$343370 : 231186 = 2^0 : 1^0.\,20', \text{ nearly.}$$

3. We have $\therefore \psi = 41^0 + 1^0.\ 20'$. Since this value differs but very little from the true one, as the following calculation shews, calculate, in the second place, on the assumption $\psi = 42^0.\ 21'$.

$\psi = 42^0.\ 20'$.	$\psi = 42^0.\ 21'$.
$log. \frac{\pi\psi}{180^0} = 0{\cdot}8685598 - 1$	$log. \frac{\pi\psi}{180^0} = 0{\cdot}8687308 - 1$
$log. Cos. \psi = 9{\cdot}8687851 - 10$	$log. Cos. \psi = 9{\cdot}8686700 - 10$
Error 2253	Error $-$ 608
Subtract $-$ 608	

2861 Difference of errors.

Again, put

$$2861 : 2253 = 1' : 47'', \text{ nearly.}$$

We obtain $\therefore \psi = 42^0.\ 20'.\ 47''$, and in this value the seconds are perfectly right, of which we can be convinced, by assuming ψ greater or less by $1''$, and making the trial with it. However, the approximation cannot be continued further, for then it would be necessary to make use of logarithms with

more than seven decimal places. Euler, in the above-cited work, by these means finds

$$\psi = 42^\circ.\ 20'.\ 47''.\ 14'''.$$

From this value of ψ we further obtain

$$\phi = \psi + 90^0 = 132^\circ.\ 20'.\ 47''.\ 14'''.$$

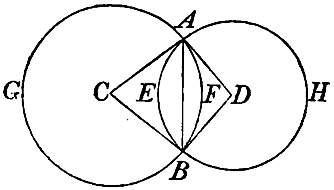
REMARK. Kästner treats this problem in a similar but more general way (Geometrical Treatise, 2nd Collection, p. 129, &c.), for he gives a method how to find a segment having a given proportion to the area of the circle.

SECTION LXV.

PROB. *Two circles intersect each other; the radii of these circles, together with the line which joins the points of section, are given: find the area of the lune which is common to both.*

SOLUT. The two circles $AFBG$, $AEBH$, (*fig.* 76) intersect each other in the points A, B; the radii $AC = r$, $AD = \rho$, together with $AB = a$ are given: find the area of the space included by the two arcs AFB, AEB, or, as it is called, the lune $AFBEA$.

fig. 76.



Put $\angle\ ACB = \phi$, $\angle\ ADB = \psi$; then, by § LXI, $Sin.\ \frac{1}{2}\phi = \frac{a}{2r}$, $Sin.\ \frac{1}{2}\psi = \frac{a}{2\rho}$. Having from hence determined the angles ϕ, ψ, then

$$\text{Segment } AFBA = \frac{\pi r^2\, Sin.\ \phi}{360^0} - \tfrac{1}{2} r^2\, Sin.\ \phi,$$

$$\text{Segment } AEBA = \frac{\pi \rho^2\, Sin.\ \psi}{360^0} - \tfrac{1}{2} \rho^2\, Sin.\ \psi.$$

As the lune $AFBEA$ consists of these two segments, consequently this last is found.

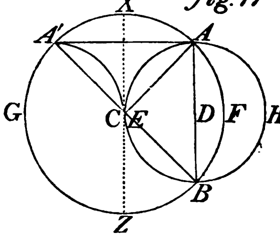
EXAM. Let $r = 27'$, $\rho = 14'$, $a = 9'.\ 5''.\ 4'''$. Here

$\phi = 20^\circ.\ 21'.\ 4''.7$, $\psi = 39^\circ.\ 50'.\ 26''\cdot9$; consequently, Segment $AFBA = 2\cdot705\ \Box'$, Segment $AEBA = 5\cdot361\ \Box'$, and $\therefore\ AFBEA = 8\cdot066\ \Box'$.

COR. When the lune $AFBEA$ has been found, the areas of the lunes $AFBHA$, $AEBGA$ may also be easily found; for it is only necessary to subtract it from the circles $AFBG$, $AEBH$.

If in any circle $AFBG$ (*fig.* 77), the two radii AC, CB, are perpendicular to one another, and AB be assumed as the diameter of the second circle $AEBH$; then $\phi = 90^\circ$, $\psi = 180^\circ$, $AB = a = r\sqrt{2}$, $\rho = \frac{1}{2}AB = r\sqrt{\frac{1}{2}}$. For this case $\therefore$

fig. 77



$$\text{Segment } AFBA = \frac{\pi r^2 \,.\, 90^\circ}{360^\circ} - \tfrac{1}{2} r^2 \, Sin. 90^\circ = \tfrac{1}{4}\pi r^2 - \tfrac{1}{2} r^2,$$

$$\text{Segment } AEBA = \frac{\frac{1}{2}\pi r^2 \,.\, 180^\circ}{360^\circ} - \tfrac{1}{4} r^2 \, Sin.\ 180^\circ = \tfrac{1}{4}\pi r^2.$$

Consequently the lune $AFBEA = \frac{1}{2}\pi r^2 - \frac{1}{2}r^2$. If we subtract this from the area $AEBH = \frac{1}{2}\pi r^2$, we then have the lune $AFBHA = \frac{1}{2}r^2 = AD^2 = \triangle\ ACB$. The area of this last lune does not depend consequently on the quadrature of the circle: thus it is always equal to the square of the radius of the lesser circle, or also equal to the triangle ACB.

Since ACB is a right angle, then the circle $AEBH$ passes through the center C of the other circle. If now the diameter XZ is drawn parallel to AB, then there are two curvilinear segments ACX, BCZ, the first of which is inclosed by the arcs AC, AX, and the radius CX; the second by the arcs BC, BZ, and the radius CZ. The area of both segments together is found by subtracting the lune $AFBEA = \frac{1}{2}\pi r^2 - \frac{1}{2}r^2$ from the semicircle $XFZ = \frac{1}{2}\pi r^2$. Consequently each of these segments is half of the triangle ACB, or half of the lune $AFBHA$.

Let the curvilinear sector XCA' be equal to the curvilinear sector XCA; then the curvilinear $\triangle\, ACA'$, inclosed by the circular arcs AA', CA, CA', is equal to the curvilinear $\triangle\, ACA' = \triangle\, ACB$. Further, ACB is a right angle, and $\therefore\ \angle\, ACX = \angle\, BCZ = \frac{1}{2} R$; consequently, if the straight lines CA', AA' are drawn, ACA' is also a right angle, and $\triangle\, ACA' = \triangle\, ACB$; $\therefore$ also the curvilinear $\triangle\, ACA' = \triangle\, ACA'$. Since the arcs AA', AC' are quadrants of their respective circles; then $Arc.\ AA' : Arc.\ AC = r : \rho$, and $\therefore\ (Arc.\ AA')^2 : (Arc.\ AC)^2 = r^2 : \rho^2 = 2 : 1$. Hence it follows, that $(Arc.\ AA')^2 = (Arc.\ AC)^2 + (Arc.\ A'C)^2$. The area of the rectilinear triangle ACA' is $\therefore$ equal to that of the curvilinear triangle ACA', and they have this common property, that the square of one of their sides is equal to the sum of the squares of the other two.

VII. PROBLEMS, WITH THEIR ANALYTICAL AND GEOMETRICAL SOLUTIONS, CHIEFLY WITH RESPECT TO GEOMETRICAL CONSTRUCTIONS.

SECTION LXVI.

PROB. *The base of a triangle, one of its angles, and the difference of the other two sides are given: find the triangle.*

SOLUT. In the triangle ABC (*fig.* 78, 79), the base BC, the angle ACB, and the difference of the two sides AC, AB are given: find the triangle. Let $BC = a$, $ACB = \alpha$, $AC - AB = \pm d$; the upper sign obtains for *fig.* 78, the lower for *fig.* 79.

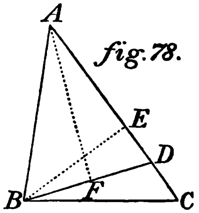


fig. 78.

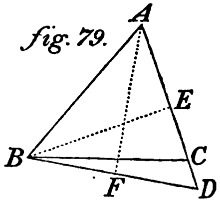


fig. 79.

1. If the angle BAC be known; then in the triangle ABC we have two angles and one side; consequently the triangle itself. Put $\therefore BAC = \phi$.

2. Then

$$AC = \frac{a \, Sin. \, (\alpha + \phi)}{Sin. \, \phi} \quad AB = \frac{a \, Sin. \, \alpha}{Sin. \, \phi}:$$

consequently, since $AC - AB = \pm d$,

$$\frac{a \, [Sin. \, (\alpha + \phi) - Sin. \, \alpha]}{Sin. \, \phi} = \pm d.$$

3. But $Sin. \, (\alpha + \phi) - Sin. \, \alpha = 2 \, Cos. \, (\alpha + \frac{1}{2} \phi) \, Sin. \, \frac{1}{2} \phi$ and $Sin. \, \phi = 2 \, Sin. \, \frac{1}{2} \phi \, Cos. \, \frac{1}{2} \phi$; we have $\therefore$

$$\frac{2\ a\ Cos.\ (\alpha+\frac{1}{2}\phi)\ Sin.\ \frac{1}{2}\phi}{2\ Sin.\ \frac{1}{2}\ \phi\ Cos.\ \frac{1}{2}\ \phi} = \pm\ d,$$

or
$$\frac{a\ Cos.\ (\alpha + \frac{1}{2}\phi)}{Cos.\ \frac{1}{2}\ \phi} = \pm\ d.$$

4. $Cos.\ (\alpha + \frac{1}{2}\phi) = Cos.\ \alpha\ Cos.\ \frac{1}{2}\ \phi - Sin.\ \alpha\ Sin.\ \frac{1}{2}\ \phi$, and $\frac{Sin.\ \frac{1}{2}\ \phi}{Cos.\ \frac{1}{2}\ \phi} = Tan.\ \frac{1}{2}\ \phi$; consequently

$$a\ (Cos.\ \alpha - Sin.\ \alpha\ Tan.\ \frac{1}{2}\ \phi) = \pm\ d,$$

and $\therefore\ Tan.\ \frac{1}{2}\ \phi = \frac{a\ Cos.\ \alpha \mp d}{a\ Sin.\ \alpha} = Cot.\ \alpha \mp \frac{d}{a\ Sin.\ \alpha}$.

Exam. When $a = 173'$, $d = -27'$, $\alpha = 56^0.\ 25'.\ 13''$, then $\phi = 80^0.\ 48'.\ 36''$.

Const. To the given base BC, apply the angle $BCA = \alpha$; on CA (*fig.* 78), when $CA > AB$, and consequently d is positive, or on CA produced (*fig.* 79), when $CA < AB$, and consequently d is negative, take $CD = d$, and draw BD; then make the angle $ABD = ADB$, and produce the lines, till they intersect each other in A, then ABC is the triangle sought.

Upon AC, BD, draw the perpendiculars BE, AF: then (because $BC = a$, $BCA = \alpha$), $BE = a\ Sin.\ \alpha$, $CE = a Cos.\ \alpha$; $\therefore\ DE = a\ Cos.\ \alpha \mp d$, (the upper sign for *fig.* 78, and the lower for *fig.* 79) and

$$Tan.\ DBE = \frac{DE}{BE} = \frac{a\ Cos.\ \alpha \mp d}{a\ Sin.\ \alpha}.$$

Now, since $\angle\ ABD = \angle\ ADB$, by the construction, consequently ADB is an isosceles triangle, and $\therefore$, because AF is perpendicular to the base BD, $DAF = \frac{1}{2}\ BAD = \frac{1}{2}\ \phi$. But $\angle\ DAF = \angle\ DBE$, because $\triangle\ ADF$ is similar to $\triangle\ DBE$; consequently also,

$$Tan.\ \frac{1}{2}\ \phi = \frac{a\ Cos.\ \alpha \mp d}{a\ Sin.\ \alpha}.$$

Synthetic Proof. The triangle ABC has the given base BC, and the given angle BCA (by the construction).

Further, in the isosceles triangle $AB = AD$, and $\therefore$

$$AC - AB = AC - AD = \pm CD = \pm d.$$

REMARK. If d is positive, and $= a$ *Cos.* ϕ, the formula gives *Tan.* $\frac{1}{2}\phi = 0$, and $\therefore \phi = 0$; in the figure the point D will then be in E, and we have $\angle ABD = \angle ADB = 90°$, consequently $AB \parallel AC$. If $d > a$ *Cos.* ϕ; then *Tan.* $\frac{1}{2}\phi$ is negative, consequently $\frac{1}{2}\phi$ is an obtuse angle, and $\phi > 180°$; then, in the figure, D is above E, $\therefore ACB$, and consequently also ABC is an obtuse angle, and in this case the lines AB, AC, do not meet on this, but on the other side of the line BC. Since $\therefore$ for this case, the angle ACB is no longer in the triangle ACB, but is an exterior angle of this figure; consequently this solution, at least, cannot be made use of here.

SECTION LXVII.

PROB. *The base of a triangle, the sum of its other two sides, and vertical angle are given: find the triangle.*

SOLUT. In the triangle BAC (*fig.* 80), the base $BC = a$, the sum of the sides $AB + AC = s$, and the vertical angle $BAC = \alpha$, are given: find the triangle.

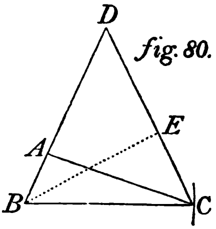


fig. 80.

1. Since $BAC = \alpha$, $ABC + ACB = 180^0 - \alpha$. The sum of the two angles at the base is consequently known, and it is only necessary to know their difference, in order to find the angles themselves. Let $\therefore ABC - ACB = \phi$; then

$$ABC = 90^0 - \tfrac{1}{2}\alpha + \tfrac{1}{2}\phi,\ ACB = 90^0 - \tfrac{1}{2}\alpha - \tfrac{1}{2}\phi.$$

2. We $\therefore$ have

$$AC = \frac{a \text{ Sin. } ABC}{\text{Sin. } \alpha} = \frac{a \text{ Cos. } \frac{1}{2}(\alpha - \phi)}{\text{Sin. } \alpha}.$$

$$AB = \frac{a \text{ Sin. } ACB}{\text{Sin. } \alpha} = \frac{a \text{ Cos. } \frac{1}{2}(\alpha + \phi)}{\text{Sin. } \alpha}.$$

Now since $AB + AC = s$,

$$\frac{a\left[\text{Cos. } \frac{1}{2}(\alpha + \phi) + \text{Cos. } \frac{1}{2}(\alpha - \phi)\right]}{\text{Sin. } \alpha} = s.$$

3. *Cos.* $\frac{1}{2}(\alpha+\phi)$ + *Cos.* $\frac{1}{2}(\alpha-\phi) = 2$ *Cos* $\frac{1}{2}\alpha$ *Cos.* $\frac{1}{2}\phi$, and *Sin.* $\alpha = 2$ *Sin.* $\frac{1}{2}\alpha$ *Cos.* $\frac{1}{2}\alpha$, consequently

$$\frac{2\ a\ Cos.\ \frac{1}{2}\ \alpha\ Cos.\ \frac{1}{2}\ \phi}{2\ Sin.\ \frac{1}{2}\ \alpha\ Cos.\ \frac{1}{2}\ \alpha} = s,$$

and $$Cos.\ \frac{1}{2}\ \phi = \frac{s\ Sin.\ \frac{1}{2}\ \alpha}{a}.$$

4. Having from hence determined ϕ; then we also have the angles $ABC = 90^0 - \frac{1}{2}(\alpha - \phi)$, $ACB = 90^0 - \frac{1}{2}(\alpha + \phi)$, and consequently also the triangle itself.

EXAM. For $a = 125'{\cdot}67$, $s = 152'{\cdot}39$, $\alpha = 49^0.\ 37'.\ 48''$, then $\phi = 118^0.\ 48'.\ 56''$, then $ABC = 124^0.\ 35'.\ 34''$, $ACB = 5^0.\ 46'.\ 38''$.

CONST. Draw a line $BD = s$; to D apply the angle $BDC = \frac{1}{2}\alpha$; from B, with a radius $BC = a$, describe a circle, which cuts the line DC in C; make the angle $DCA = ADC$, and draw CB; then ABC is the triangle sought.

Upon DC draw the perpendicular BE; then $BE = BD\,Sin.\,BDC = s\ Sin.\ \frac{1}{2}\ \alpha$, and $Cos.\ CBE = \frac{BE}{BC} = \frac{s\ Sin.\ \frac{1}{2}\ \alpha}{a} = Cos.\ \frac{1}{2}\ \phi$; consequently, $CBE = \frac{1}{2}\ \phi$, $BCE = 90^0 - \frac{1}{2}\ \phi$. We $\therefore$ have $ACB = DCB - DCA = 90^0 - \frac{1}{2}\phi - \frac{1}{2}\alpha$, $ABC = DBE + EBC = 90^0 - \frac{1}{2}\alpha + \frac{1}{2}\ \phi$, which was required.

Synthetic Proof. Since $\angle\ BDC = \angle\ DCA$; $BAC = BDC + DCA = 2\ BDC = \alpha$; also $BA + AC = BA + AD = BD = s$, and $BC = a$; consequently BAC is the required triangle.

REMARK. The problem is only positive, when $a < s$, and also $s\ Sin.\ \frac{1}{2}\ \alpha < a$: the first, because the sum of two sides of a triangle is always greater than the third side; the second, because $Cos.\ \frac{1}{2}\ \phi$ cannot be greater than 1. It is also evident from the figure, that $BC = a$ cannot be greater than $BE = s\ Sin.\ \frac{1}{2}\ \alpha$, because otherwise it could not reach the line CD, which is required by the construction.

If in the problem, the difference of the sides, instead of the sum, be given; then it may be solved in a similar way.

SECTION LXVIII.

Prob. *The base of a triangle, the difference of the two angles at the base, and the difference of the sides, are given: find the triangle.*

Solut. In the triangle ABC (*fig.* 81), the base $BC = a$, the difference of the angles at the base, or $ABC - ACB = \alpha$, and the difference of the two sides, or $AC - AB = d$, are given: find the triangle.

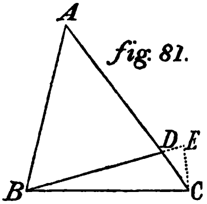


fig. 81.

1. Since the difference of the angles ABC, ACB is given; consequently it is only necessary to know their sum, in order to be able to determine the angles themselves. Put $\therefore$ $ABC + ACB = \phi$; then we have, $ABC = \frac{1}{2}(\phi + \alpha)$, $ACB = \frac{1}{2}(\phi - \alpha)$; further, $BAC = 180^0 - \phi$.

2. Hence it follows, that

$$AC = \frac{a \; Sin. \frac{1}{2}(\phi + \alpha)}{Sin. \; \phi}, \quad AB = \frac{a \; Sin. \frac{1}{2}(\phi - \alpha)}{Sin. \; \phi}.$$

Therefore $AC - AB = \dfrac{a[Sin. \frac{1}{2}(\phi + \alpha) - Sin. \frac{1}{2}(\phi - \alpha)]}{Sin. \; \phi} = d.$

3. $Sin. \frac{1}{2}(\phi + \alpha) - Sin. \frac{1}{2}(\phi - \alpha) = 2 \; Cos. \frac{1}{2}\phi \; Sin. \frac{1}{2}\alpha$, $Sin. \; \phi = 2 \; Sin. \frac{1}{2}\phi \; Cos. \frac{1}{2}\phi$; consequently,

$$\frac{2 \; a \; Cos. \frac{1}{2}\phi \; Sin. \frac{1}{2}\alpha}{2 \; Sin. \frac{1}{2}\phi \; Cos \frac{1}{2}\phi} = d.$$

and $\therefore$ $$Sin. \frac{1}{2}\phi = \frac{a \; Sin. \frac{1}{2}\alpha}{d}.$$

4. Having from hence determined the value of ϕ; we then likewise have the angles $ABC = \frac{1}{2}(\phi + \alpha)$, $ACB = \frac{1}{2}(\phi - \alpha)$, $BAC = 180^0 - \phi$, and $\therefore$, since the line BC is also given, the triangle ABC is known.

Exam. When $a = 234'$, $d = 98'$, $\alpha = 30^0. \; 59'. \; 34''$,

then $\frac{1}{2}\phi = 39^0.\ 38'.\ 22''$, $ABC = \frac{1}{2}\phi + \frac{1}{2}\alpha = 55^0.\ 8'.\ 9''$, $ACB = \frac{1}{2}\phi - \frac{1}{2}\alpha = 24^0.\ 8'.\ 35''$.

Const. To BC, the given base of the triangle, apply the angle $CBD = \frac{1}{2}\alpha$, and from C, with a distance $CD = d$, cut the line BD in D; draw CD, and produce it; then make the angle $DBA = ADB$, and produce BA, DA, till they meet in A: then ABC is the triangle sought.

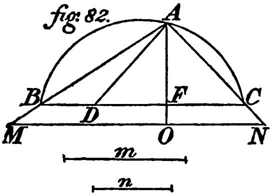
Produce BD, and draw CE perpendicular to it: then $CE = BC\ Sin.\ CBE = a\ Sin.\ \frac{1}{2}\ \alpha$, $Sin.\ CDE = \frac{CE}{CD}$ $= \frac{a\ Sin.\ \frac{1}{2}\alpha}{d} = Sin\ \frac{1}{2}\phi$; consequently $CDE = ADB = ABD$ $= \frac{1}{2}\phi$, $ABC = ABD + CBD = \frac{1}{2}\phi + \frac{1}{2}\alpha$, and $ACB = ADB$ $- CBD = \frac{1}{2}\phi - \frac{1}{2}\alpha$, which was required.

Remark. The problem is impossible, when CD is less than the perpendicular CE; for then the line CD cannot reach BD. This agrees also with the calculation, since d cannot be less than $a\ Sin.\ \frac{1}{2}\ \alpha$, because otherwise $Sin.\ \frac{1}{2}\ \phi > 1$, which is impossible.

SECTION LXIX.

Prob. *The vertical angle of a triangle, and the segments into which the perpendicular drawn from the vertex of this angle divides the base, are given: find the triangle.*

Solut. In the triangle ABC (*fig.* 82), the vertical angle $BAC = \alpha$ is given; also the segments $BF = a$, $CF = b$, into which the base is divided by the perpendicular AF, are given: determine the triangle.



1. Since $BF = a =$ $AF\ Tan.\ BAF$, and $CF = b = AF\ Tan.\ CAF$: therefore

$$a : b = Tan.\ BAF : Tan.\ CAF;$$

consequently

$$a+b:a-b=Tan.BAF+Tan.CAF:Tan.BAF-Tan.CAF$$
$$=Sin.\ (BAF+CAF):Sin.\ (BAF-CAF)$$
$$=Sin.\ \alpha : Sin.\ (BAF-CAF).$$

2. Hence we obtain

$$Sin.\ (BAF-CAF)=\frac{(a-b)\ Sin.\ \alpha}{a+b}$$

or, since $BAF=90^0-ABC$, and $CAF=90^0-ACB$, and $\therefore BAF-CAF=ACB-ABC$,

$$Sin.\ (ACB-ABC)=\frac{(a-b)\ Sin.\ \alpha}{a+b}$$

3. Having from this equation determined the difference $ACB-ABC$ of the angles at the base; then, because their sum is already known, we have also the angles ACB, ABC, for $ACB+ABC=180^0-\alpha$.

Exam. When $a=247'$, $b=53'$, $\alpha=113^0.\ 20'.\ 54''$, we find $ACB-ABC=36^0.\ 25'.\ 15''\cdot25$. Now since $ACB+ABC=66^0.39'.6''$: therefore $ACB=51^0.32'.10''\cdot62$, $ABC=15^0.\ 6'.\ 55''\cdot37$.

Const. On an indefinite line, make $BF=a$, $CF=b$; upon BC describe a segment of a circle BAC, containing an angle equal to the given angle α. From F draw the perpendicular FA, which meets the segment in A, and draw the lines BA, CA; then BAC is the required triangle.

Make $FD=FC$; then $\angle FAD=\angle FAC$, $\angle ADC=\angle ACD$; further

$$BC:AB=Sin.\ BAC:Sin.\ ACB$$
$$AB:BD=Sin.\ ADB:Sin.\ BAD$$
$$=Sin.\ ACB:Sin.\ BAD,$$

consequently $BC:BD=Sin.\ BAC:Sin.\ BAD$;
or since $BC=a+b$, $BD=a-b$, $BAC=\alpha$, $BAD=BAF-FAD=BAF-FAC=ACB-ABC$,

$$a+b:a-b=Sin.\ \alpha:Sin.\ (ACB-ABC)$$

and $\therefore$ $Sin. (ACB - ABC) = \frac{(a - b) \, Sin. \, \alpha}{a + b}$,

which was required.

Cor. If the segments themselves are not given, but merely their proportion to one another, and also the altitude of the required triangle: let $m : n$ be the given proportion between the segments, and p the altitude of the triangle. First construct, as was done before, a triangle BAC, whose segments BF, FC are equal to the lines m, n, and upon the perpendicular AF, produced if necessary, make $AO = p$; through O, parallel to BC, draw the line MN, which cuts AB, AC, or these lines produced, in M, N: the $\triangle\, MAN$ is the required triangle. This method is very easily understood.

If instead of the altitude of the triangle, one of its sides, viz. AM, be given: then in AB, or this line produced, it is only necessary to determine the point M, so that AM may be of the required length, and then from M to draw MN parallel to BC, in order to obtain the required triangle MAN.

SECTION LXX.

Prob. *The vertical angle of a triangle, the sum of its sides, and the difference of the segments into which a perpendicular from the vertex of this angle divides the base, are given: find the triangle.*

Solut. 1. Let BAC (*fig.* 82) be the required triangle, the sum of its sides $AB + AC = a$, the difference of the segments $BF - CF = d$, and the vertical angle $BAC = \alpha$. If the difference of the sides AB, AC be also known; then we can determine each of these. Assume therefore $AB - AC = x$: then $AB = \frac{a + x}{2}$, $AC = \frac{a - x}{2}$. Further, let $BC = y$.

2. Since $AB^2 = AF^2 + BF^2$, $AC^2 = AF^2 + CF^2$: then $AB^2 - AC^2 = BF^2 - CF^2$, or

$$(AB + AC)(AB - AC) = (BF + CF)(BF - CF),$$

or

$$ax = dy.$$

3. In the triangle BAC, we have

$$BC^2 = AB^2 + AC^2 - 2\,AB\,.\,AC\,.\,Cos.\,\alpha,$$

or $y^2 = \left(\frac{a+x}{2}\right)^2 + \left(\frac{a-x}{2}\right)^2 - \frac{a^2-x^2}{2}\,Cos.\,\alpha.$

If in this last equation we substitute for y its value $\frac{ax}{d}$ taken from 2, and solve the equation, we then obtain

$$x = \sqrt{\frac{a^2d^2(1 - Cos.\,\alpha)}{2\,a^2 - d^2(1 + Cos.\,\alpha)}}$$

or, since $1 + Cos.\,\alpha = 2\,Cos.^2\,\frac{1}{2}\alpha$, $1 - Cos.\,\alpha = 2\,Sin.^2\,\frac{1}{2}\alpha$,

$$x = \sqrt{\frac{a^2d^2\,Sin.^2\,\frac{1}{2}\alpha}{a^2 - d^2\,Cos.^2\,\frac{1}{2}\alpha}} = \frac{ad\,Sin.\,\frac{1}{2}\alpha}{\sqrt{(a^2 - d^2\,Cos.^2\frac{1}{2}\alpha)}}.$$

Solut. 2. 1 From 2 of the foregoing solution, we have

$$a(AB - AC) = dy,$$

and $\therefore$ $AB - AC = \frac{dy}{a}.$

2. Assume the angle $ABC = \phi$; then $ACB = 180^0 - (\alpha + \phi)$, and

$$AB = \frac{y\,Sin.\,(\alpha + \phi)}{Sin.\,\alpha},\ AC = \frac{y\,Sin.\,\phi}{Sin.\,\alpha};$$

consequently, $AB - AC = \frac{y\,[Sin.\,(\alpha + \phi) - Sin.\,\phi]}{Sin.\,\alpha}$

$$= \frac{2\,y\,Cos.\,(\frac{1}{2}\alpha + \phi)\,Sin.\,\frac{1}{2}\alpha}{2\,Sin.\,\frac{1}{2}\alpha\,Cos\,\frac{1}{2}\alpha},$$

$$= \frac{y\,Cos.\,(\frac{1}{2}\alpha + \phi)}{Cos.\,\frac{1}{2}\alpha}$$

3. If the two expressions for $AB - AC$ in 1 and 2 be put equal to one another, we then obtain

$$Cos.\,(\tfrac{1}{2}\alpha + \phi) = \frac{d\,Cos.\,\frac{1}{2}\alpha}{a};$$

whence the angle ϕ may be determined.

4. Further, since $AB+AC=a$; therefore from 2 we have

$$\frac{y\,[Sin.\ (\alpha+\phi)+Sin.\ \phi]}{Sin.\ \alpha}=a,$$

or
$$\frac{2\,y\,Sin.\ (\tfrac{1}{2}\,\alpha+\phi)\ Cos.\ \tfrac{1}{2}\,\alpha}{2\ Sin.\ \tfrac{1}{2}\,\alpha\ Cos.\ \tfrac{1}{2}\,\alpha}=a,$$

and $\therefore$
$$y=\frac{a\ Sin.\ \tfrac{1}{2}\,\alpha}{Sin.\ (\tfrac{1}{2}\,\alpha+\phi)};$$

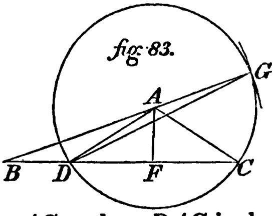
further,
$$AB=\frac{y\ Sin.\ (\alpha+\phi)}{Sin.\ \alpha}=\frac{a\ Sin.\ (\alpha+\phi)}{2\ Cos.\ \tfrac{1}{2}\,\alpha\ Sin.\ (\tfrac{1}{2}\alpha+\phi)},$$

$$AC=\frac{y\ Sin.\ \phi}{Sin.\ \alpha}=\frac{a\ Sin.\ \phi}{2\ Cos.\ \tfrac{1}{2}\,\alpha\ Sin.\ (\tfrac{1}{2}\,\alpha+\phi)}.$$

Since the second solution gives all the parts of the required triangle at once, by means of formulæ, which are simple and easy of calculation, it is consequently preferable to the first.

Exam. When $a=207'$, $d=13'.\ 7''$, $\alpha=61^0.\ 23'.\ 18''$, we find $\tfrac{1}{2}\,\alpha+\phi=86^0.\ 44'.\ 14''\cdot8$, $\phi=56^0.\ 2'.\ 35''\cdot8$; and hence $BC=y=105'\cdot8357$, $AB=107'\cdot0022$, $AC=99'\cdot9976$.

Const. Upon an indefinite line (*fig.* 83), take $BD=d$, and make CDG equal to half the adjacent angle of the given vertical angle, consequently $=90^0-\tfrac{1}{2}\,\alpha$; from B, with the radius $BG=a$, cut the line DG in G, and join BG; then make the angle $GDA=DGA$, and from A, with the radius AB, describe a circle, which cuts the indefinite line in C, and join AC; then BAC is the required triangle.



For since $CDG=90^0-\tfrac{1}{2}\alpha$; therefore $BGD=90^0-\tfrac{1}{2}\,\alpha-ABC$; consequently $Sin.BDG=Sin.\ CDG=Cos.\tfrac{1}{2}\,\alpha$, $Sin.\ BGD=Sin.\ [90^0-(\tfrac{1}{2}\alpha+ABC)]=Cos.\ (\tfrac{1}{2}\,\alpha+ABC)$. But in the triangle BGD

$$BG:BD=Sin.\ BDG:Sin.\ BGD,$$

or $\quad a : d = Cos. \frac{1}{2} \alpha : Cos. (\frac{1}{2} \alpha + ABC)$;

consequently $Cos. (\frac{1}{2} \alpha + ABC) = \dfrac{d \; Cos. \frac{1}{2} \alpha}{a}$.

But analytically

$$Cos. (\tfrac{1}{2} \alpha + \phi) = \frac{d \; Cos. \frac{1}{2} \alpha}{a};$$

we have $\therefore \frac{1}{2} \alpha + ABC = \frac{1}{2} \alpha + \phi$, and $ABC = \phi$, which were required.

Synthetic Proof. Upon BC draw the perpendicular AF; then BF, FC, are the segments of the triangle BAC.

Since $\angle GAC = 2 \angle GDC$, (*Euc. III.* 20); therefore GAC is the adjacent angle of the vertical angle in the required triangle; consequently BAC is this vertical angle. Further

$$AB + AC = AB + AG = BG = a,$$
$$BF - CF = BF - DF = BD = d.$$

Q. E. D.

SECTION LXXI.

PROB. *The base of a triangle, its altitude, and the difference of the angles at the base, are given: find the triangle.*

SOLUT. Let the required triangle be BAC (*fig.* 82); the base $BC = a$, the altitude $AF = h$, and the difference of the angles at the base $ACB - ABC = \alpha$.

1. If from these data the sum of the angles ACB, ABC, can be determined; we then have the angles, and consequently also the triangle. Put $\therefore ACB + ABC = \phi$; then $ACB = \frac{1}{2} (\phi + \alpha)$, $ABC = \frac{1}{2} (\phi - \alpha)$.

2. From the right-angled triangles AFC, AFB, we obtain

$$CF = h \; Cot. \tfrac{1}{2} (\phi + \alpha), \; BF = h \; Cot. \tfrac{1}{2} (\phi - \alpha).$$

Now since $CF + BF = a$; we have the equation

$$h \left[Cot. \tfrac{1}{2} (\phi + \alpha) + Cot. \tfrac{1}{2} (\phi - \alpha) \right] = a.$$

3. But $Cot. \frac{1}{2}(\phi + \alpha) + Cot. \frac{1}{2}(\phi - \alpha)$

$$= \frac{Sin.\ \phi}{Sin.\ \frac{1}{2}(\phi + \alpha)\ Sin.\ \frac{1}{2}(\phi - \alpha)},$$

$$Sin.\ \tfrac{1}{2}(\phi + \alpha)\ Sin.\ \tfrac{1}{2}(\phi - \alpha) = \tfrac{1}{2}\left[Cos.\ \alpha - Cos.\ \phi\right];$$

instead of the former equation, we consequently have the following one:

$$\frac{h\ Sin.\ \phi}{\frac{1}{2}\left[Cos.\ \alpha - Cos.\ \phi\right]} = a,$$

or
$$Sin.\ \phi + \frac{a}{2h}\ Cos.\ \phi = \frac{a}{2h}\ Cos.\ \alpha.$$

4. In order from hence to determine the value of ϕ in the easiest way, we can make use of the following artifice. Put $Tan.\ \mu$ for $\frac{a}{2h}$; consequently determine an angle μ such, that its tangent is equal in numerical value to the expression, $\frac{a}{2h}$; by these means the foregoing equation is transformed into the following one:

$$Sin.\ \phi + Tan.\ \mu\ Cos.\ \phi = Tan.\ \mu\ Cos.\ \alpha;$$

or, when we multiply both sides by $Cos.\ \mu$, and substitute $Sin.\ \mu$ for $Cos.\ \mu\ Tan.\ \mu$,

$$Cos.\ \mu\ Sin.\ \phi + Sin.\ \mu\ Cos.\ \phi = Sin.\ \mu\ Cos.\ \alpha,$$

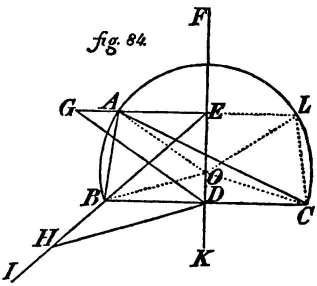
or likewise $\quad Sin.\ (\phi + \mu) = Sin.\ \mu\ Cos.\ \alpha,$

whence ϕ may now be determined.

Exam. Let $a = 1365'$, $h = 789'$, $\alpha = 21^0.\ 39'.\ 18''$. Here $log.\ Tan.\ \mu = log.\ \frac{a}{2h} = 9{\cdot}9370257$, and $\therefore \mu = 40^0.\ 51'.\ 37''{\cdot}55$. Hence we obtain further, $log.\ Sin.\ (\phi + \mu)$ $log.\ Sin.\ \mu + log.\ Cos.\ \alpha = 9{\cdot}7839362$, and consequently $\phi + \mu = 37^0.\ 26'.\ 53''\ 96$, or $\phi + \mu = 142^0.\ 33'.\ 6''{\cdot}04$. The first of the two values found for $\phi + \mu$ need not to be used here, because $\phi + \mu$ must necessarily be greater than μ. If $\therefore$ we take the second value, we then obtain, $\phi = 101^0.\ 41'.\ 28''{\cdot}49$, and hence $ACB = \frac{1}{2}(\phi + \alpha) = 61^0.\ 40'.\ 23''{\cdot}24$, $ABC = \frac{1}{2}(\phi - \alpha) = 40^0.\ 1'.\ 5''{\cdot}24$.

CONST. From D, the centre of the given base BC, (*fig.* 84), draw KF perpendicular to it, and make $DE=h$; from E draw the line EG parallel to BC, and to ED apply the angle $EDG=\alpha$; then by these means the point G is determined. Draw EB, and from D, with the distance DG, cut EB produced in H; then describe upon BC an arc, including the angle HDK; and from the point A, in which this arc cuts the line EG, draw the lines AB, AC; then BAC is the triangle sought.

fig. 84



This construction may be derived from the analytical solution, in the following way:

Since $BD=\frac{1}{2}a$, $DE=h$; therefore $Tan.\ BED=\dfrac{BD}{DE}=\dfrac{a}{2h}=Tan.\ \mu$; consequently $BED=\mu$. Further, in the triangle HED,

$$HD : ED = Sin.\ HED : Sin.\ DHE,$$

or since $HD=DG=DE\ Sec.\ GDE=\dfrac{h}{Cos.\ \alpha}$,

$$\frac{h}{Cos.\ \alpha} : h = Sin.\ \mu : Sin.\ DHE,$$

consequently $Sin.\ DHE=Sin.\ DHI=Sin.\ \mu\ Cos.\ \alpha$.

But by the analytical solution also, $Sin.\ (\phi+\mu)=Sin.\ \mu+Cos.\ \alpha$; we have $\therefore$ either $\phi+\mu=DHE$, or $\phi+\mu=DHI$. The first supposition cannot obtain here; for since $DH=DG>DE$, consequently also $\angle DEH>\angle DHE$ (*Euc. I.* 18); $\therefore \mu>\phi+\mu$, which is impossible. We $\therefore$ have $DHI=\phi+\mu$, and consequently $EDH=DHI-DEH=\phi$. Now since ϕ denotes the sum of the two angles at the base of the required triangle; consequently HDK must be the vertical angle of this triangle; from which all the rest necessarily follows.

Synthetic Proof. The triangle BAC, as appears immediately

from the construction, has the given base and altitude; $\therefore$ it only remains to be proved, that likewise $ABC - ACB = GDE = a$. Produce $\therefore$ the line GE, till it cuts the circular arc in L, and from the centre O draw the radii OA, OB, OC, OL: then $BOC = 2\,BAC$ (*Euc. III.* 20), and $BOD = \frac{1}{2}BOC = BAC$; but likewise $HDK = BAC$ (construction), consequently $HDK = BOK$, and $BO \parallel HD$. Further, since $DH = DG$, $OB = OA$; consequently $DH : OB = DG : OA$; but likewise $DH : OB = DE : EO$, consequently $DG : OA = DE : EO$, $\triangle AEO$ is similar to $\triangle GED$ (*Euc. VI.* 7), and $\therefore OA \parallel DG$. Now draw CL, then $ACL = BCL - ACB = ABC - ACB$; further, $ACL = \frac{1}{2}AOL = AOE = GDE$; consequently $ABC - ACB = GDE$.

Q. E. D.

SECTION LXXII.

PROB. *To divide a straight line, so that the rectangle contained by the two parts may be equal to a given square.*

SOLUT. The given line $AB = a$ (*fig.* 85) is required to be divided in G, so that $AG \times GB = mn^2 = b^2$.

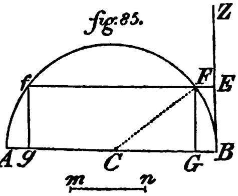


fig. 85.

Let $AG = x$; then $BG = a - x$; rectangle $AG \times GB = x(a-x)$, and $\therefore$

$$x(a-x) = b^2.$$

The solution of this equation gives

$$x = \frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2 - b^2)}.$$

CONST. Upon AB describe a semicircle AFB: from B draw the perpendicular BZ, and then make $BE = mn$; from E draw the line EF parallel to AB, which cuts the semicircle in F, f; and from these points draw FG, fg, perpendicular to AB: then G or g is the required point of section.

Draw the radius CF: then $CG^2 = CF^2 - FG^2 = CB^2 - BE^2 = \frac{1}{4}a^2 - b^2$; consequently $CG = Cg = \sqrt{(\frac{1}{4}a^2 - b^2)}$, and $\therefore AG = \frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 - b^2)}$. $AG = \frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 - b^2)}$, which are the two values found for x.

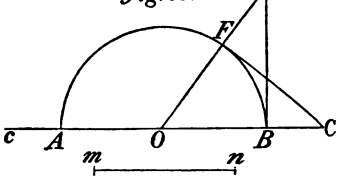
The synthetic proof of this construction is founded on *Euc. VI.* 13, 17.

SECTION LXXIII.

PROB. *To produce a given line, so that the rectangle contained by the whole line thus produced, and the part produced, may be equal to the square of a given line.*

SOLUT. Let the line AB be given (*fig.* 86): lengthen it by the part BC, so that the rectangle $AC \times CB = mn^2$

fig. 86.



Make $AB = a$, $mn = b$, $BC = x$; then $AC = a + x$, and we ∴ have the following equation:

$$x(a + x) = b^2.$$

The solution gives

$$x = -\tfrac{1}{2}a \pm \surd(\tfrac{1}{4}a^2 + b^2.)$$

CONST. Upon AB describe a semicircle; from B draw the perpendicular $BE = mn$; and from the centre O draw the line OE; to the point F, in which this line cuts the semicircle, draw the tangent FC, which meets AB produced in C; then BC is the required part, by which AB must be produced.

For since, as is readily seen, $\triangle\ EOB$ is similar and equal to $\triangle\ FOC$, and $OE^2 = OB^2 + BE^2 = \frac{1}{4}a^2 + b^2$: then $OC = OE = \surd(\frac{1}{4}a^2 + b^2)$, and ∴

$$BC = OC - OB = \surd(\tfrac{1}{4}a^2 + b^2) - \tfrac{1}{2}a.$$

If we produce the line AB in the other direction, and make $Oc = OC$, we then have $Bc = OB + OC = \frac{1}{2}a + \surd(\frac{1}{4}a^2 + b^2)$, which is the absolute value of this line. But with respect to the position, if Bc be opposite to BC, we ∴ have analytically $Bc = -\frac{1}{2}a - \surd(\frac{1}{4}a^2 + b^2)$, which is the second value of x. Since in the problem nothing is said as to which side the line AB is to be produced on, consequently both values of x verify it, because $Bc\,.\,cA = BC\,.\,CA$. But if it is expressed in the problem, that the

line is to be produced towards B, then the first value of x need only be retained.

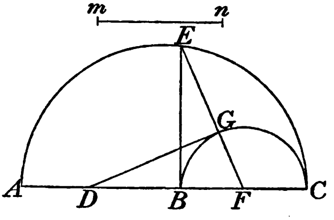
The synthetic proof of the construction is founded on *Euc. III.* 36, and is easily deduced from it.

SECTION LXXIV.

Prob. *To divide a given line into two parts, so that the rectangle contained by one part and another given line, may be equal to the square of the other part.*

Solut. Let (*fig.* 87) two lines AB, mn, be given: divide the first in D, so that rectangle $AD \times mn = BD^2$.

fig. 87.



Let $AB = a$, $mn = b$, $BD = x$: then $AD = a - x$, and $\therefore$, from the conditions of the problem, we have the equation

$$(a - x)\, b = x^2.$$

The solution of this equation gives

$$x = -\tfrac{1}{2} b \pm \sqrt{(ab + \tfrac{1}{4} b^2)}.$$

Const. Upon that part of AB which is produced, take $BC = mn$, and upon AC, BC, describe two semicircles; from B draw the perpendicular BC, and from F, the center of the semicircle BGC, draw the line FE; to the point of intersection G, draw the tangent GD, which meets AB in D: then D is the required point of section.

For since $BE^2 = AB \times BC$ (*Euc. VI.* 13, 17) $= ab$, and $BF = \frac{1}{2} BC = \frac{1}{2} b$: consequently $FE^2 = BE^2 + BF^2 = ab + \frac{1}{4} b^2$, and $FE = \sqrt{(ab + \frac{1}{4} b^2)}$. But $\triangle\, DFG$ is similar and equal to $\triangle\, BFE$, and $\therefore$ $DF = FE$; consequently also $DF = \sqrt{(ab + \frac{1}{4} b^2)}$, and $\therefore$ $DB = DF - BF = \sqrt{(ab + \frac{1}{4} b^2)} - \frac{1}{2} b$, which was required.

The second value of x, because it is negative, does not obtain here, otherwise the point D, which, according to the

problem, is to be an intersection of the line AB, will fall beyond the point B, and consequently not between A and B.

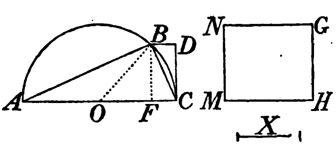
Synthetic Proof. Since $\triangle DFG$ is similar and equal to $\triangle BFE$: then $BE = DG$. Now $BE^2 = AB \times BC$ (*Euc. VI.* 13, 17), $DG^2 = BD \times DC$ (*Euc. III.* 36); consequently also $AB \times BC = BD \times DC$. Take the rectangle $BD \times BC$ from both sides; we then have $(AB - BD)\, BC = (DC - BC) \times BD$, or $AD \times BC = AD \times mn = BD^2$. Q. E. D.

SECTION LXXV.

PROB. *To find two lines such, that the sum of their squares may be equal to a given square, together with the rectangle contained by these lines.*

SOLUT. Let AC (*fig.* 88) the side of the given square $= a$; let the given rectangle be MG, its sides $MN = b$, $NG = c$. Let x, y be the two lines sought. From the conditions of the problem, we obtain the two following equations:

fig. 88.



$$xy = bc,\ x^2 + y^2 = a^2.$$

The solution of these equations gives:

$$x = \sqrt{\left[\tfrac{1}{2}a^2 + \sqrt{\left(\tfrac{1}{4}a^4 - b^2c^2\right)}\right]}$$
$$y = \sqrt{\left[\tfrac{1}{2}a^2 - \sqrt{\left(\tfrac{1}{4}a^4 - b^2c^2\right)}\right]}$$

Hence we get the construction.

CONST. Find the fourth proportional to the three lines AC, MN, NG (*Euc. VI.* 12); let this be X, so that $AC : MN = NG : X$. Then describe upon AC a semicircle, and from C draw the perpendicular $CD = X$; from D draw DB parallel to AC, which meets the semicircle in B, and from B draw the lines BA, BC; these are the lines sought.

Draw the radius OB, and the perpendicular BF. Since $AC : MN = NG : X$ (Construction), or $a : b = c : X$; therefore $X = CD = BF = \frac{bc}{a}$: consequently $OF = \sqrt{(OB^2 - BF^2)}$ $= \sqrt{\left(\frac{1}{4}a^2 - \frac{b^2c^2}{a^2}\right)}$. Hence we obtain further: $AF = \frac{1}{2}a + \sqrt{\left(\frac{1}{4}a^2 - \frac{b^2c^2}{a^2}\right)}$, $CF = \frac{1}{2}a - \sqrt{\left(\frac{1}{4}a^2 - \frac{b^2c^2}{a^2}\right)}$. Now since $AB^2 = AC \times AF = \frac{1}{2}a^2 + a\sqrt{\left(\frac{1}{4}a^2 - \frac{b^2c^2}{a^2}\right)} = \frac{1}{2}a^2 + \sqrt{(\frac{1}{4}a^4 - b^2c^2)}$, $BC^2 = AC \times CF = \frac{1}{2}a^2 - a \times \sqrt{\left(\frac{1}{4}a^2 - \frac{b^2c^2}{a^2}\right)} = \frac{1}{2}a^2 - \sqrt{(\frac{1}{4}a^4 - b^2c^2)}$: consequently $AB = \sqrt{[\frac{1}{2}a^2 + \sqrt{(\frac{1}{4}a^4 - b^2c^2)}]}$, $BC = \sqrt{[\frac{1}{2}a^2 - \sqrt{(\frac{1}{4}a^4 - b^2c^2)}]}$, which was required

Synthetic Proof. Since ABC is a right angle, consequently $AC^2 = AB^2 + BC^2$, which verifies the first condition of the problem. Further, since $AC : MN = NG : X$ $(= BF)$: consequently rectangle $AC \times BF =$ rectangle $MN \times NG$. But rectangle $AC \times BF =$ rectangle $AB \times BC$, because both rectangles are double the triangle ABC; consequently also rectangle $AB \times BC =$ rectangle $MN \times NG$. Q. E. D.

REMARK. We may also, as I presume to be already known, give the analytical expressions for x and y, the following more simple forms:

$$x = \tfrac{1}{2}[\sqrt{(a^2 + 2bc)} + \sqrt{(a^2 - 2bc)}]$$
$$y = \tfrac{1}{2}[\sqrt{(a^2 + 2bc)} - \sqrt{(a^2 - 2bc)}].$$

The formulæ given in the solution are, however, better adapted to the construction.

SECTION LXXVI.

PROB. *To find a right-angled triangle such, that its hypothenuse is equal to a given line, and the rectangle contained by its two sides is equal to the square of the difference of these two sides.*

SOLUT. Let ABC (*fig.* 88) be the required triangle,

whose hypothenuse $AC = a$ is given, and in which rectangle $AB \times BC = (AB - BC)^2$.

Let $AB = x$, $BC = y$: then, from the conditions of the problem, we obtain the two following equations:

$$x^2 + y^2 = a^2, \; xy = (x - y)^2;$$

and these give

$$x = \sqrt{\frac{a^2}{6}(3 + \sqrt{5})}, \; y = \sqrt{\frac{a^2}{6}(3 - \sqrt{5})}.$$

In order to adapt these expressions found for x and y to the construction, give them the following forms:

$$\sqrt{a[\tfrac{1}{2}a + \sqrt{(\tfrac{1}{4}a^2 - \tfrac{1}{9}a^2)}]}, \; \sqrt{a[\tfrac{1}{2}a - \sqrt{(\tfrac{1}{4}a^2 - \tfrac{1}{9}a^2)}]}.$$

Const. Upon a given hypothenuse $AC = a$, describe a semicircle; from C draw the perpendicular $CD = \frac{1}{3}AC = \frac{1}{3}a$, and from D draw DB parallel to AC, which meets the semicircle in B; from B draw the lines BA, BC: ABC is the triangle sought.

Draw the radius OB, and the perpendicular BF; then $BF = CD = \frac{1}{3}a$, $OB = \frac{1}{2}a$, consequently $OF = \sqrt{(OB^2 - BF^2)} = \sqrt{(\frac{1}{4}a^2 - \frac{1}{9}a^2)}$, and $\therefore$ $AF = \frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 - \frac{1}{9}a^2)}$, $CF = \frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 - \frac{1}{9}a^2)}$. Now since $AB^2 = AC \times AF$, $BC^2 = AC \times CF$; consequently $AB = \sqrt{a[\frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 - \frac{1}{9}a^2)}]}$, $BC = \sqrt{a[\frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 - \frac{1}{9}a^2)}]}$, which was required.

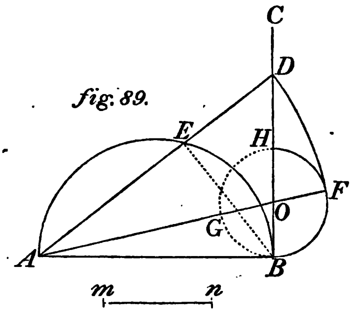
Synthetic Proof. Since $\triangle ABC$ is similar to $\triangle BFC$: then $AB : AC = BF : BC$; consequently also $3\,AB : AC = 3\,BF : BC = AC : BC$, and $\therefore$ $3\,AB \times BC = AC^2$. But because ABC is a right angle, $AC^2 = AB^2 + BC^2$; we consequently have $3\,AB \times BC = AB^2 + BC^2$, and $AB \times BC = AB^2 - 2\,AB \times BC + BC^2 = (AB - BC)^2$. Q. E. D.

SECTION LXXVII.

Prob. *At the extremity of the given diameter of a semicircle, a perpendicular is drawn: find a point in this*

perpendicular such, that when a line is drawn from this point to the other extremity of the diameter, that part of it which is without the circle, may be equal to a given line.

Solut. Let AB (*fig.* 89) be the given diameter of a semicircle, BC a perpendicular upon it: find a point D such, that when DA is drawn, DE may be equal to a given line mn.



Let $AB = a$, $mn = b$, $AD = x$, and draw BE. Since AEB is a right angle, consequently $\triangle\ ADB$ is similar to $\triangle\ ABE$, and $\therefore$

$$AD : AB = AB : AE$$

$$x : a = a : x - b.$$

Hence we obtain

$$x^2 - bx = a^2,$$

and $$x = \tfrac{1}{2}b + \sqrt{(\tfrac{1}{4}b^2 + a^2)}.$$

Const. Upon BC, make $BH = mn = b$, and upon BH describe a semicircle BFH; from A through its center O draw the line AF, and with a radius AF, describe an arc, which cuts the line BC in D: D is the required point.

For since $AB = a$, $BO = \frac{1}{2}BH = \frac{1}{2}b$; consequently $AO = \sqrt{(BO^2 + AB^2)} = \sqrt{(\frac{1}{4}b^2 + a^2)}$; then $AD = AF = AO + OF = \frac{1}{2}b + \sqrt{(\frac{1}{4}b^2 + a^2)}$, which was required.

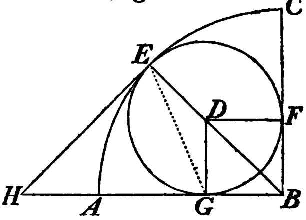
Synthetic Proof. Complete the circle $BFHG$. Since BD touches the circle AEB, and AB the circle $BFHG$; $\therefore BD^2 = AD \times DE$, $AB^2 = FA \times AG$; consequently $AD \times DE + FA \times AG = BD^2 + AB^2 = AD^2$, and $\therefore FA \times AG = AD^2 - AD \times DE = AD \times AE$. Now $FA = AD$; consequently also $AG = AE$, and $\therefore DE = GF$, or, since $GF = BH = mn$, $DE = mn$. Q. E. D.

SECTION LXXVIII.

PROB. *In a given quadrant to describe a circle which touches both the circumference and the two radii.*

SOLUT. Let ACB (*fig.* 90) be the given quadrant, and its radius $= a$. Further, let EFG be the required circle, D its centre, and E, F, G, the points in which it touches the circumference, and the two radii of the quadrant.

fig. 90.



1. Draw DF, DG: then DFB, DGB, are right angles (*Euc.* III. 18); likewise ABC is a right angle; consequently $BFDG$ is a parallelogram, and since $DG = DF$, it is also a square; ∴ DBC is half a right angle. Produce BD; consequently this line passes through the point E. (*Euc.* III. 2).

2. Let $DE = DF = DG = x$; then $BD = \sqrt{(BG^2 + DG^2)} = \sqrt{2x^2} = x\sqrt{2}$; consequently $BE = BD + DE = x\sqrt{2} + x$. Now, since also $BE = a$: therefore

$$x\sqrt{2} + x = a,$$

and ∴ $$x = \frac{a}{\sqrt{2} + 1} = a(\sqrt{2} - 1),$$

or likewise $$x = \sqrt{2a^2} - a.$$

Hence we obtain the following construction.

CONST. Bisect the angle ABC by the line BE; to the point E, where this line meets the circumference AC, draw the tangent EH, which meets BA produced in H; then make $BG = AH$, and from G draw the perpendicular GD; from D, where this perpendicular meets BE, with a radius DG, describe the circle EFG: this is the required circle.

For since BEH is a right angle, and EBH half a right angle; consequently $BE = EH = a$, and ∴ $BH =$

$BH = \sqrt{(BE^2 + EH^2)} = \sqrt{2\,a^2}$; consequently $BG = AH = BH - BA = \sqrt{2\,a^2} - a$, which was required.

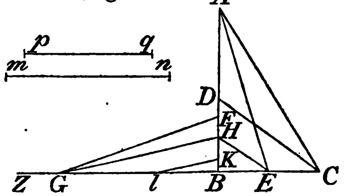
Synthetic Proof. Draw DF perpendicular to BC; then $DFB = DGB$, $DBG = DBF$, $BD = BD$; consequently $\triangle\ DGB$ is similar and equal to $\triangle\ DFB$, and $DF = DG$; hence the circle described with the radius DG passes through F, and touches the lines BC, BA, in F, G (*Euc. III.* 16). Further, since $HBE = \frac{1}{2}R$: then $BE = EH$; but likewise $AH = BG$ (Construction), consequently $HG = AB = BE = EH$, and $\therefore$ $\triangle\ HGE$ is isosceles, and $HGE = HEG$. But also $DEH = DGH\ (= R)$; consequently $DEG = DGE$, and $\therefore$ $DG = DE$. Therefore the circle described with the radius DG, also passes through E, and it meets the circumference AB in no other point, otherwise a line drawn from B to this point would be equal to BE, which is impossible. (*Euc. III.* 8).

SECTION LXXIX.

PROB. *The lines drawn from the vertical angle of a right-angled triangle to the centre of the opposite sides, are given: find the triangle.*

SOLUT. Find a right-angled triangle ABC (*fig.* 91) such, that the sides AB, BC, containing the right angle, when bisected in D, E, and the lines AE, CD are drawn, $AE = mn = a$, and $CD = pq = b$.

fig. 91.



Put $BD = x$: then $BC^2 = CD^2 - BD^2 = b^2 - x^2$; consequently $BE^2 = \frac{1}{4}\,BC^2 = \frac{b^2 - x^2}{4}$, and $AE^2 = AB^2 + BE^2 = 4x^2 + \frac{b^2 - x^2}{4} = \frac{b^2 + 15\,x^2}{4}$. Now since $AE = a$; we consequently have the equation

$$\frac{b^2 + 15\,x^2}{4} = a^2,$$

and hence $\quad x = \sqrt{\dfrac{4\,a^2 - b^2}{15}}.$

CONST. Upon any line AB, to the point B draw a perpendicular CZ, and make $BF = \frac{1}{2}pq = \frac{1}{2}b$; from F with a distance $FG = mn = a$ cut BZ in G; in BA take any part BK, from K, with a distance $Kl = 4\,BK$, cut BZ in l; and from G draw GH parallel to Kl, which meets the line BA in H; then from H make $HE = \frac{1}{2}pq = \frac{1}{2}b$ cut the line BC in E, and make $BC = 2\,BE$, $BA = 4\,BH$. Draw AC: then ABC is the triangle sought.

For since $FG = a$, $BF = \frac{1}{2}b$ (Construction): then $BG^2 = FG^2 - BF^2 = a^2 - \frac{1}{4}b^2$. Further, because $\triangle\ BHG$ is similar to $\triangle\ BKI$, $GH : BH = Kl : KB = 4 : 1$ (Construction), and $\therefore$ $GH = 4\,BH$, $GH^2 = 16\,BH^2$, $BG^2 = GH^2 - BH^2 = 15\ BH^2$; consequently $15\,BH^2 = a^2 - \frac{1}{4}b^2$, $BH^2 = \dfrac{a^2 - \frac{1}{4}b^2}{15}$; $\therefore$ (because $x = \frac{1}{2}\,AB = 2\,BH$),

$x^2 = \dfrac{4\,a^2 - b^2}{15}$, and $x = \sqrt{\dfrac{4\,a^2 - b^2}{15}}$, which was required.

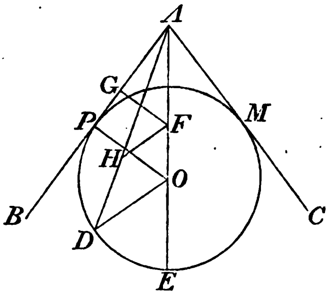
Synthetic Proof. Assume $BD = 2\,BH = \frac{1}{2}AB$, and draw CD: then, because also $BC = 2\,BE$, $CD \parallel HE$, and $\therefore$ $CD = 2\,HE = pq$ (Construction); wherefore the line CD, which is drawn from C to the centre of AB, is of the given length. Further, because the triangles BKI, BGH, are similar, as was proved before, $GH = 4\,BH$, and also $AB = 4\,BH$ (Construction): then $AB = GH$, and $\therefore$ $GB^2 = GH^2 - BH^2 = AB^2 - BH^2$. But $AE^2 - HE^2 = AB^2 - BH^2$, (because $AE^2 = AB^2 + BE^2$, $HE^2 = BH^2 + BE^2$), consequently $GB^2 = AE^2 - HE^2$. But likewise $GB^2 = GF^2 - BF^2 = GF^2 - HE^2$, (because by the Construction, $BF = \frac{1}{2}pq = HE$); consequently we have $AE^2 - HE^2 = GF^2 - HE^2$, and $\therefore$ $AE = GF$; hence, since $GF = mn$ (Construction), also $AE = mn$. Q. E. D.

SECTION LXXX.

PROB. *An angle and a point in it are given: describe a circle which touches both the lines containing the angle, and passes through the given point.*

SOLUT. Let the given angle be BAC (*fig.* 92), the given point D, PMD the required circle, which touches the lines AB, AC, in P, M, and passes through the point D.

fig. 92.



1. Through O, the centre of the circle, draw the line AE; consequently this bisects the angle BAC. Further, since the point D is given, we likewise have the angle DAE, and the line AD. Therefore let $BAE = \alpha$, $DAE = \beta$, $AD = a$.

2. If the angle ADO be known; then in the triangle ADO we have one side and two angles, consequently also the lines AO, OD, and at the same time the centre and radius of the circle. However, this angle can be very easily found; for since $AO : OP = AO : OD$, and $AO : OP = 1 : Sin.\ \alpha$, $AO : OD = Sin.\ ADO : Sin.\ \beta$: then

$$1 : Sin.\ \alpha = Sin.\ ADO : Sin.\ \beta,$$

and $\therefore$ $Sin.\ ADO = \dfrac{Sin.\ \beta}{Sin.\ \alpha}$.

Hence we get the following very simple Construction.

CONST. Bisect the angle BAC by the line AE, and then take any point F; from this point draw FG perpendicular to AC, and with the distance $FH = FG$, cut the line AD in H: then from D draw DO parallel to HF: O is the centre of the circle.

For since $FH = FG$ (Construction); therefore $AF : FG =$

$AF : FH$. But $AF : FG = 1 : Sin.\,\alpha$, $AF : FH = Sin.\,AHF : Sin.\,\beta$; consequently $1 : Sin.\,\alpha = Sin.\,AHF : Sin.\,\beta$, and $\therefore\ Sin.\,AHF = Sin.\,ADO = \frac{Sin.\,\beta}{Sin.\,\alpha}$, which was required.

Synthetic Proof. Draw OP perpendicular to AB: then $OP \parallel FG$, $DO \parallel HF$; consequently $AF : FG = AO : OP$, and $AF : FH = AO : OD$. But $AF : FG = AF : FH$, (because $FG = FH$, by the construction); consequently $AO : OP = AO : OD$, and $\therefore\ OP = OD$. A circle described with a radius OD, will consequently pass through P, and at this point touch the line AB, consequently also AC.

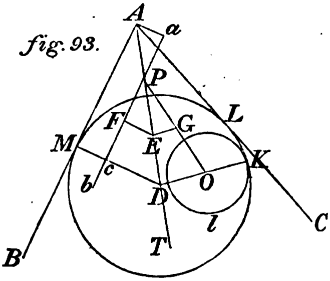
REMARK. Since there is another point besides H in the line AD, from which $FH = FG$: consequently strictly there are two circles, which verify the problem,

SECTION LXXXI.

PROB. *An angle, and a circle within it, are given: describe another circle, which at the same time touches the two lines containing the given angle and the given circle.*

SOLUT. Let BAC (*fig.* 93) be the given angle, Kl the given circle, whose centre is O, and MLK the required circle, which touches the lines AB, AC in M, L, and the circle in K. The centre D of the required circle must, as in the preceding section, be in the line AD, which bisects the angle BAC; further, the centres of the two circles, and the point of contact, must be in a straight line (*Euc.* *III.* 11,)

fig. 93.



1. In AD take any point P, and draw PO: then, besides

the angle BAC, and the radius OK of the given circle, the lines AP, PO, and the angle OPD may be considered as given. Therefore let $DAC = \frac{1}{2} BAC = \alpha$, $OPD = \beta$, $AP = a$, $PO = b$, $OK = r$.

2. If the angle POD be known; then we can draw the line OD, and then the centre D is determined. Assume ∴ $POD = \phi$: then

$$DO = \frac{b \, Sin. \, \beta}{Sin. \, (\beta + \phi)}, \quad PD = \frac{b \, Sin. \, \phi}{Sin. \, (\beta + \phi)},$$

consequently $DK = DO + OK = \dfrac{b \, Sin. \, \beta}{Sin. \, (\beta + \phi)} + r,$

$$AD = PD + AP = \frac{b \, Sin. \, \phi}{Sin. \, (\beta + \phi)} + a.$$

3. Now $DK = DM = AD \, Sin. \, \alpha$; we consequently have the equation

$$\frac{b \, Sin. \, \beta}{Sin. \, (\beta + \phi)} + r = \left[\frac{b \, Sin. \, \phi}{Sin. \, (\beta + \phi)} + a\right] Sin. \, \alpha,$$

or

$$b \, Sin. \beta + r \, Sin. (\beta + \phi) = b \, Sin. \alpha \, Sin. \phi + a \, Sin. \alpha \, Sin. (\beta + \phi):$$

from which equation the value of ϕ may be determined.

4. Since the point P may be assumed arbitrarily: determine it ∴, so that the obtained equation may be more simple. This is the case, when we put $AP = a = \dfrac{r}{Sin. \, \alpha}$; for by these means the foregoing equation is transformed into the following one:

$$Sin. \, \beta = Sin. \, \alpha \, Sin. \, \phi,$$

and this gives $Sin. \, \phi = \dfrac{Sin. \, \beta}{Sin. \, \alpha}$;

from which we obtain a very easy construction.

Const. Bisect the given angle BAC by the line AT; from A draw the perpendicular $Aa = r$, make ab parallel to

AB, and from the point P, where it cuts the line AT, draw PO to the centre of the given circle. In AT take any point E, draw EF perpendicular to ab, and with the distance $EG = EF$, cut the line PO in G. Through O draw the line DK parallel to EG; then the point D, in which this line meets AT, is the centre, and DK the radius of the required circle.

For we have $AP = \dfrac{Aa}{Sin.\ APa} = \dfrac{r}{Sin.\ \alpha}$, as was required. Further, because $EF = EG$ (Construction), $EF : EP = EG : EP$; but $EF : EP = Sin.\ \alpha : 1$, $EG : EP = Sin.\ \beta : Sin. PGE$; consequently $Sin. \alpha : 1 = Sin. \beta : Sin. PGE$, and $\therefore$ $Sin.\ PGE = Sin.\ POD = \dfrac{Sin.\ \beta}{Sin.\ \alpha}$.

Synthetic Proof. Draw DM perpendicular to AB: then $Dca = DMA = R$; consequently, since also $EFP = R$ (Construction) $Dc \parallel EF$. Now likewise $DO \parallel EG$, hence $PE : EF = PD : Dc$, and $PE : EG = PD : DO$. But $PE : EF = PE : EG$ (because $EF = EG$), consequently $PD : Dc = PD : DO$, and $\therefore$ $Dc = DO$; and since also $cM = Aa = OK$; then $DM = DK$. A circle described with the radius DK, passes $\therefore$ through M, and touches the line AB, consequently also AC. But since the two circles touch in K, it appears, from hence, that they can have no other common point, because otherwise a line drawn from D to this point must be equal to DK, which is impossible *(Euc. III. 8.)*

REMARK. Since in PO there are always two points, such as G, from which $EG = EF$; consequently there are also always, as in the foregoing section, two circles, which verify the problem.

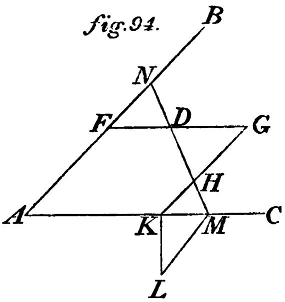
SECTION LXXXII.

PROB. *An angle and a point within it are given: through this point draw a line, which meets the two lines including the angle, and with them forms a triangle of a given area.*

SOLUT. Let BAC *(fig. 94)* be the given angle, and D

the given point within it: through this point draw a line MN such, that the $\triangle NAM$ has a given area.

fig. 94.



1. Through D draw a line FG parallel to AC: then the lines AF, FD may be considered as known. To AF apply a parallelogram $AFGK$, which has the given area (*Euc. I.* 45): consequently the solution of the problem depends merely upon this, to draw the line MN in such a way, that $\triangle NAM$ = parallelogram $AFGK$.

2. If this be done: then $\triangle DGH = \triangle DNF + \triangle HKM$. But $\triangle DGH : \triangle DNF = DG^2 : DF^2$, and $\triangle DGH : \triangle HKM = DG^2 : KM^2$ (*Euc. VI.* 19); consequently also $\triangle DGH : \triangle DNF + \triangle HKM = DG^2 : DF^2 + KM^2$, and since $\triangle DGH = \triangle DNF + \triangle HKM$, then likewise $DG^2 = DF^2 + KM^2$, and $\therefore$

$$KM = \sqrt{(DG^2 - DF^2)}.$$

Hence, since the lines DG, DF are known, the line KM, and at the same time also the point M, may be very easily determined, both by calculation and by notation.

Const. After having applied a parallelogram $AFGK$ to AF, as required in the solution, upon AC draw from K the perpendicular KL, and with the distance $LM = DG$, cut the line KC in M: then from M through D draw the line MN: what was required is now done.

The proof readily follows from the solution itself.

SECTION LXXXIII.

Prob. *Two lines intersecting each other at right angles, and a point which is equally distant from these two lines are given: through this point draw a line, so*

that the part of it contained between the two lines containing one of the right angles may be equal to a given line.

SOLUT. Let the two lines Bb, Cc (*fig.* 95) intersect each other at right angles in the point A; let the point D be so situated, that the two perpendiculars DP, Dp are equal to one another; through D draw a line DM, so that its part EM, which lies between the lines Ab, Ac, may be equal to the given line XZ.

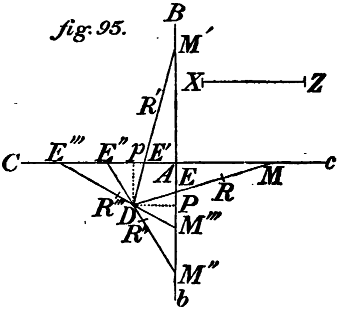


fig. 95.

1. If the line DE or DM be determined: then the problem is solved. Instead, however, of finding these lines immediately, it will be better, to assume their sum for the unknown magnitude in the calculation, because their difference is already given. Let $\therefore DP = Dp = a$, $DM - DE = XZ = 2b$, $DM + DE = 2x$: then $DM = x + b$, $DE = x - b$.

2. Since $\triangle DPE$ is similar to $\triangle EAM$: therefore

$$DE : DP = EM : AM$$

or

$$x - b : a = 2b : AM;$$

and $\therefore$

$$AM = \frac{2ab}{x - b},$$

$$Mp = a + \frac{2ab}{x - b} = \frac{a(x + b)}{x - b}.$$

3. Now $DM^2 = Mp^2 + Dp^2$; we $\therefore$ have the equation

$$(x + b)^2 = a^2 + \frac{a^2 (x + b)^2}{(x - b)^2},$$

or

$$x^4 - 2(a^2 + b^2)x^2 + b^4 - 2a^2b^2 = 0.$$

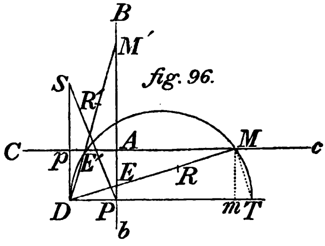
This bi-quadratic equation is a quadratic one for x^2; its

solution gives

$$x = \pm \sqrt{[a^2 + b^2 \pm a \sqrt{(a^2 + 4 b^2)}]}.$$

4. Consequently x has four values, all of which verify the equation. For since it was not expressed in the problem, in which of the four right angles the part EM between the lines containing the angles was situated; consequently this line may have four different situations; viz. either the position EM, within the angle bAc, as was assumed in the solution, or the position $E'M'$, within the angle BAC, or even the two positions $E''M''$, $E'''M'''$, within the angle bAC. If the line EM be bisected in R: then $DR = \frac{1}{2}(DM + DE) = x$; likewise if the lines $E'M'$, $E''M''$, $E'''M'''$, be bisected in R', R'', R''': then DR', DR'', DR''', are the three other values of x. The lines DR, DR', are equal with respect to their absolute magnitudes, and only differ in their positions; these are expressed by the two values $\pm \sqrt{[a^2 + b^2 + a \sqrt{(a^2 + 4b^2)}]}$. In like manner, the lines DR'', DR''', only differ in their position, and are expressed by the two values $\pm \sqrt{[a^2 + b^2 - a \sqrt{(a^2 + 4 b^2)}]}$. The two first values of x, which correspond to the lines DR, DR', are always positive; the two last are only so, when $a^2 + b^2 > a \sqrt{(a^2 + 4b^2)}$, or $a^4 + 2 a^2 b^2 + b^4 > a^4 + 4 a^2 b^2$, that is $b^4 > 2 a^2 b^2$, and $b^2 > 2 a^2$.

Const. Let Bb, Cc, (*fig.* 96) be the two lines intersecting each other, D the given point, and $DP = Dp = a$, be two perpendiculars to the former of these lines. In Dp take the part $DS = XZ$ (preceding figure) $= 2b$, and draw PS; then in DP produced, take $PT = PS$; upon DT describe a semicircle, which cuts the line Cc in M, and draw DM: then the part EM, which lies between the two lines including the angle bAc, is equal to the given line XZ.



For since $DP = a$, $DS = 2b$: then $PS = \sqrt{(a^2 + 4b^2)}$, and $\therefore$ $DT = DP + PT = DP + PS = a + \sqrt{(a^2 + 4b^2)}$. Now draw MT, and the perpendicular Mm. Since $\angle TMm$

$= \angle MDT$, and $DP = Mm$: consequently the right-angled triangles TMm, DPE, are equal, and $\therefore MT = DE$. If EM be bisected in R: then $DR = \frac{1}{2}(DM + DE) = \frac{1}{2}(DM + MT)$. Now $DM^2 + MT^2 = DT^2$, and $DM \times MT = Mm \times DT = a . DT$, hence $DM^2 + 2DM . MT + MT^2 = DT^2 + 2a . DT = DT(2a + DT) = [a + \sqrt{(a^2 + 4b^2)}] \times [3a + \sqrt{(a^2 + 4b^2)}] = 4a^2 + 4b^2 + 4a\sqrt{(a^2 + 4b^2)}$, and $\therefore DM + MT = \sqrt{[4a^2 + 4b^2 + 4a\sqrt{(a^2 + 4b^2)}]}$, $DR = \sqrt{[a^2 + b^2 + a\sqrt{(a^2 + 4b^2)}]}$, as was required.

Synthetic Proof. Since $EM^2 = AM^2 + AE^2$, and $AP^2 = (AE + EP)^2 = AE^2 + 2AE \times EP + EP^2$: therefore $AM^2 + 2AE^2 + 2AE \times EP + EP^2 = EM^2 + AP^2$, or $AM^2 + 2PA \times AE + EP^2 = EM^2 + DP^2$. In the similar triangles AEM, DEP, $DP : AM = PE : AE$; and $DP \times AE\,(= PA \times AE) = AM \times EP$. We therefore have $AM^2 + 2AM \times EP + EP^2 = EM^2 + DP^2$, or $(AM + EP)^2 = EM^2 + DP^2$; consequently since $AM = Pm$, and $EP = mT$, $(Pm + mT)^2 = PT^2 = EM^2 + DP^2$. But likewise $PT^2 = PS^2 = DS^2 + DP^2$; hence $EM^2 + DP^2 = DS^2 + DP^2$, $\therefore EM^2 = DS^2$, and $EM = DS$. Q. E. D.

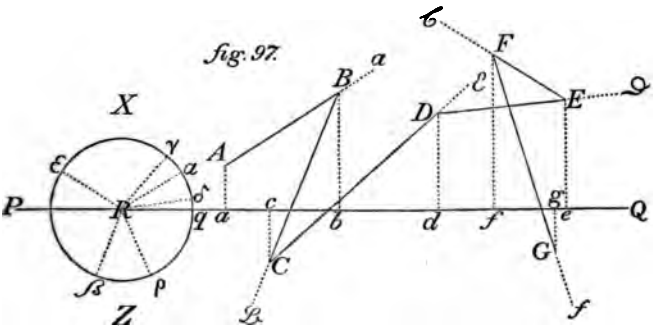
REMARK. The circle cuts the line Cc twice in E'. If the line DM' be drawn through this point, which meets Bb in M': then, if EM' be bisected in R', DR' is the second value of x denoted by the same letters as in *fig.* 95, and consequently $= DR$. This is very easily proved. For the other two values of x, a similar construction may be found, which, however, I shall not give here, in order that I may have room for other matter.

VIII. POLYGONOMETRICAL PROBLEMS.

SECTION LXXXIV.

DEFINITIONS.

1. If from all the angular points and extremities of a crooked line $ABCDEF$ (*fig.* 97), the perpendiculars Aa,



Bb, Cc, Dd, Ee, Ff, Gg, be let fall upon any given straight line PQ: these are called Ordinates. Further, if in the line PQ we assume any point R; then the parts of this line which lie between this point and the ordinates, are called Abscissæ; thus Ra is the abscissa of the point A, Rb the abscissa of the point B, and so on: further, PQ is the axis of abscissæ, R the origin of abscissæ; the abscissa and ordinate of a point form together the co-ordinates of this point.

If the co-ordinates of a point are given; consequently also the point itself is. But the ordinate of a point may fall on both sides of the line PQ: consequently the determination

of the absolute magnitude of the ordinates is not sufficient; we must also know their position. In order to distinguish the above two positions from one another, the ordinates above the line PQ are expressed by $+$; on the other hand, those which are below it, by $-$. Therefore the points A, B, C, D, E, F, G, have the ordinates $+Aa$, $+Bb$, $-Cc$, $+Dd$, $+Ee$, $+Ff$, $-Gg$. There is a similar relation between the abscissæ; thus, the abscissæ which are from R, in the direction of Q, are denoted by $+$, and those which are in the direction of P, are expressed by $-$.

2. The lines AB, BC, CD, DE, EF, FG, of which the crooked line $ABCDEFG$ is composed, are called its lines of division. But if the crooked line inclose a space; consequently these lines, as we already know, are sides of the figure thus formed. If a line of division be produced, in order to avoid mistakes, it is denoted according to the order of the alphabet. Produce ∴ (when the contrary is not expressly required) the line AB not towards A, but towards B; in like manner BC, not towards B, but towards C, and so on. In this case it is advisable to denote the crooked line, from its first to its last point, according to the letters of the alphabet.

3. When from any point R of the axis of abscissæ, which need not be exactly the beginning of the abscissa, a circle is described, and the radii $R\alpha$, $R\beta$, $R\gamma$, $R\delta$, $R\varepsilon$, $R\zeta$, are respectively drawn parallel to the lines of division AB, BC, CD, DE, EF, FG, according to the order in which they are produced: then these radii are called the corresponding radii of these lines; viz. $R\alpha$ is the corresponding radius of AB, $R\beta$ the corresponding radius of BC, and so on.

4. By the exterior angles of a crooked line are here meant those angles which the continuation of a line of division makes with the other at their point of junction; thus CBa, DCb, EDc, FEd, GFe.

It is known, that when two lines, say M, N, are parallel to two others, say P, Q, viz. $M \parallel P$, $N \parallel Q$, the angle, which the lines M, N form at their junction, is equal to the

angle included by the lines P, Q. Consequently also the exterior angle of two lines of division, is equal to the angle formed by their corresponding radii; consequently $aBb = = a R\beta$, $bCc = \beta R\gamma$, $cDd = \gamma R\delta$, $dEe = \delta R\varepsilon$, $eFf = \varepsilon R\zeta$.

The absolute magnitude of the exterior angles is not sufficient, however, to determine the situation of two lines of division, because these lines may form a convex as well as a concave angle, although the exterior angle continues to be of the same magnitude: this ∴ must be considered. With this view, imagine a moving line turning round the point R, and advancing successively in the directions $R\alpha$, $R\beta$, $R\gamma$, $R\delta$, $R\varepsilon$, $R\zeta$, which, by these means, describe the angles $\alpha R\beta = aBb$, $\beta R\gamma = bCc$, $\gamma R\delta = cDd$: then it is evident, that this radius in advancing from $R\beta$ towards $R\gamma$, must have a motion, which, as to its direction, is exactly contrary to that which it describes in moving from $R\alpha$ towards $R\beta$. If ∴ we consider the angles, which are generated by the motion of a radius from the object Q towards the objects X, P, Z, as positive; those angles which require a contrary movement may be considered as negative. Therefore the angles bCc, dEe, eFf, require the sign $+$, and the angles aBb, cDd, the sign $-$.

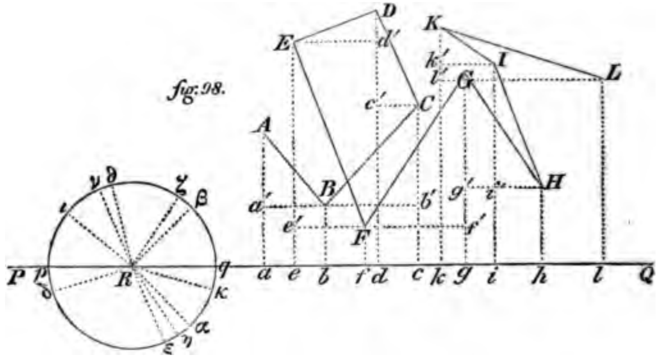
Let the exterior angles of a crooked line aBb, bCc, cDd, dEe, eFf, with their proper signs, $+$ or $-$, be constantly expressed according to the order, by the single letters B, C, D, E, F; ∴ in the foregoing figure $B = - aBb$, $C = + bCc$, $D = - cDd$, $E = dEe$, $F = eFf$. Further, let the angle which the first corresponding radius, and consequently also the first line of division, makes with the line of abscissæ, here $qR\alpha$, be denoted by A. Let the line $R\alpha$ be above the line rq, then $A = - qR\alpha$.

SECTION LXXXV.

Prob. *The lines of division of a crooked line, its exterior angles, and the co-ordinates of the first point, together with the angle, which the first corresponding radius*

makes with the line of abscissæ, are given: determine the co-ordinates of each angular point.

Solut. Let *ABCDEFGHIKL* (*fig.* 98) be the crooked



line, its lines of division $AB = a$, $BC = b$, $CD = c$, $DE = d$, $EF = e$, $FG = f$, $GH = g$, $HI = h$, $IK = i$, $KL = k$; let the corresponding radii of these lines, according to their order, be $R\alpha$, $R\beta$, $R\gamma$, $R\delta$, $R\varepsilon$. Further, let R be the origin of the abscissæ, the abscissa of the first point $Ra = p$, its ordinate $Aa = q$.

1. From the points B, C, D, E, &c. to the line of abscissæ PQ, draw the perpendiculars Bb, Cc, Dd, Ee, &c. We then have

$$
\begin{array}{llllll}
Aa & = q & & & & \\
Bb & = Aa - Aa' & = q & - a \text{ Sin. } ABa' & = q & - a \text{ Sin. } qR\alpha \\
Cc & = Bb + Cb' & = Bb & + b \text{ Sin. } CBb' & = Bb & + b \text{ Sin. } qR\beta \\
Dd & = Cc + Dc' & = Cc & + c \text{ Sin. } DCc' & = Cc & + c \text{ Sin. } pR\gamma \\
Ee & = Dd - Dd' & = Dd & - d \text{ Sin. } DEd' & = Dd & - d \text{ Sin. } pR\delta \\
Ff & = Ee - Ee' & = Ee & - e \text{ Sin. } EFe' & = Ee & - e \text{ Sin. } qR\varepsilon \\
Gg & = Ff + Gf' & = Ff & + f \text{ Sin. } GFf' & = Ff & + f \text{ Sin. } qR\zeta \\
Hh & = Gg - Gg' & = Gg & - g \text{ Sin. } GHg' & = Gg & - g \text{ Sin. } qR\eta \\
Ii & = Hh + Ii' & = Hh & + h \text{ Sin. } IHi' & = Hh & + h \text{ Sin. } pR\theta \\
Kk & = Ii + Kk' & = Ii & + i \text{ Sin. } KIk' & = Ii & + i \text{ Sin. } pR\iota \\
Ll & = Kk - Kl' & = Kk & - k \text{ Sin. } KLl' & = Kk & - k \text{ Sin. } qR\kappa,
\end{array}
$$

further:

$$
\begin{aligned}
Ra &= p \\
Rb &= Ra + ab = p + a\,Cos.\,ABa' = p + a\,Cos.\,q\,R\,\alpha \\
Rc &= Rb + bc = Rb + b\,Cos.\,CBb' = Rb + b\,Cos.\,q\,R\,\beta \\
Rd &= Rc - cd = Rc - c\,Cos.\,DCc' = Rc - c\,Cos.\,p\,R\,\gamma \\
Re &= Rd - de = Rd - d\,Cos.\,DEd' = Rd - d\,Cos.\,p\,R\,\delta \\
Rf &= Re + ef = Re + e\,Cos.\,EFe' = Re + e\,Cos.\,q\,R\,\varepsilon \\
Rg &= Rf + fg = Rf + f\,Cos.\,GFf' = Rf + f\,Cos.\,q\,R\,\zeta \\
Rh &= Rg + gh = Rg + g\,Cos.\,GHg' = Rg + g\,Cos.\,q\,R\,\eta \\
Ri &= Rh - hi = Rh - h\,Cos.\,IHi' = Rh - h\,Cos.\,p\,R\,\theta \\
Rk &= Ri - ik = Ri - i\,Cos.\,KIk' = Ri - i\,Cos.\,p\,R\,\iota \\
Rl &= Rk + kl = Rk + k\,Cos.\,KLl' = Rk + k\,Cos.\,q\,R\,\kappa.
\end{aligned}
$$

2. But

$$Sin.\,A = Sin. - q\,R\,\alpha = -\,Sin.\,q\,R\,\alpha$$

$$Sin.\,(A + B) = Sin.\,(-\,q\,R\,\alpha + \alpha\,R\,\beta) = Sin.\,q\,R\,\beta$$

$$Sin.\,(A + B + C) = Sin.\,(-\,q\,R\,\alpha + \alpha\,R\,\beta + \beta\,R\,\gamma) = Sin.\,q\,R\,\gamma = Sin.\,p\,R\,\gamma$$

$$Sin.\,(A+B+C+D) = Sin.\,(-q\,R\,\alpha+\alpha\,R\,\beta+\beta R\gamma+\gamma R\delta) = Sin.\,q\,\overset{*}{R}\,\delta\,{}^{*}) = -\,Sin.\,p\,R\,\delta$$

$$Sin.\,(A + B + C + D + E) = Sin.\left\{\begin{array}{c}-qR\alpha + \alpha R\beta + \beta R\gamma \\ + \gamma\,R\,\delta + \delta\,R\,\varepsilon\end{array}\right\} = Sin.\,q\,\overset{*}{R}\,\varepsilon = -\,Sin.\,q\,R\,\varepsilon$$

$$Sin.\,(A+B+C+D+E+F) = Sin.\left\{\begin{array}{c}-qR\alpha+\alpha R\beta+\beta R\gamma \\ +\gamma R\delta+\delta R\varepsilon+\varepsilon R\zeta\end{array}\right\} = Sin.\,(360^0 + q\,R\,\zeta) = Sin.\,q\,R\,\zeta$$

$$Sin.\,(A+B+C+\ldots+G) = Sin.\left\{\begin{array}{c}-qR\alpha+\alpha R\beta+\beta R\gamma \\ +\gamma R\delta+\delta R\varepsilon+\varepsilon R\zeta \\ -\zeta B\eta\end{array}\right\} = Sin.\,q\,\overset{*}{R}\,\eta = -\,Sin.\,q\,R\,\eta$$

* The asterisk above $Sin.\,q\,R\,\delta$ denotes, that by $q\,R\,\delta$ is to be understood not the concave, but the convex angle.

$$Sin.\ (A+B+C+\,..\,+H) = Sin.\left\{\begin{array}{c}-qR\alpha+\alpha R\beta+\beta R\gamma\\ +\gamma R\delta+\delta R\varepsilon+\varepsilon R\zeta\\ -\zeta R\eta+\eta R\theta\end{array}\right\}$$

$$= Sin.\,(360^{\circ}+q\,R\,\theta) = Sin.\,q\,R\,\theta = Sin.\,p\,R\,\theta$$

$$Sin.\,(A+B+C+\,...\,+I) = Sin.\left\{\begin{array}{c}-qR\alpha+\alpha R\beta+\beta R\gamma\\ +\gamma R\delta+\delta R\varepsilon+\varepsilon R\zeta\\ -\zeta R\theta+\eta R\theta+\theta R\iota\end{array}\right\}$$

$$= Sin.\,(360^{\circ}+q\,R\,\iota) = Sin.\ p\,R\,\iota$$

$$Sin.\ (A+B+C+\,...\,+K) = Sin.\left\{\begin{array}{c}-qR\alpha+\alpha R\beta+\beta R\gamma\\ +\gamma R\delta+\delta R\varepsilon+\varepsilon R\zeta\\ -\zeta R\eta+\eta R\theta+\theta R\iota\\ -\iota R\kappa\end{array}\right\}$$

$$= Sin.\,q\,\dot{R}\,\kappa^{*} = -\,Sin.\,q\,R\,\kappa\,;$$

and in like manner

$Cos.\ A = Cos.\,-\,q\,R\,\alpha = Cos.\ q\,R\,\alpha$

$Cos.\ (A+B) = Cos.\,q\,R\,\beta$

$Cos.\ (A+B+C) = Cos.\,q\,R\,\gamma = -\,Cos.\,p\,R\,\gamma$

$Cos.\ (A+B+C+D) = Cos.\,q\,\dot{R}\,\delta = -\,Cos.\,p\,R\,\delta$

$Cos.\ (A+B+C+D+E) = Cos.\,q\,\dot{R}\,\varepsilon = Cos.\,q\,R\,\varepsilon$

$Cos.\ (A+B+B+\ldots+F) = Cos.\,(360^{\circ}+q\,R\,\zeta) = Cos.\,q\,R\,\zeta$

$Cos.\ (A+B+C+\ldots+G) = Cos.\,q\,\dot{R}\,\eta = Cos.\,q\,R\,\eta$

$Cos.\ (A+B+C+\,..\,+H) = Cos.\,(360^{\circ}+qR\theta) = -\,Cos.\,pR\theta$

$Cos.\ (A+B+C+\ldots+I) = Cos.\,(360^{\circ}+qR\iota) = -\,Cos.\,pR\iota$

$Cos.\ (A+B+C+\ldots+K) = Cos.\,q\,\dot{R}\,\kappa = Cos.\,q\,R\,\kappa.$

3. Now, if we substitute the values of $Sin.\,q\,R\,\alpha$, $Sin.\,q\,R\,\beta$, $Sin.\,p\,R\,\gamma$, $Sin.\,p\,R\,\delta$, &c. $Cos.\,q\,R\,\alpha$, $Cos.\,q\,R\,\beta$, $Cos.\,p\,R\,\gamma$, $Cos.\,p\,R\,\delta$, &c., found in 2, in the expression in 1, for the lines Aa, Bb, Cc, &c., Ra, Rb, Rc, &c., we obtain

$Aa = q$

$Bb = q + a\,Sin.\,A$

$Cc = Bb + b\,Sin.\,(A+B) = q + a\,Sin.\,A + b\,Sin\ (A+B)$

$Dd = Cc + c\,Sin.\,(A+B+C) = q + a\,Sin.\,A + b\,Sin.\,(A+B)$
$\quad + c\,Sin.\,(A+B+C)$

$Ee = Dd + d\,Sin.\,(A+B+C+D) = q + a\,Sin.\,A$
$\quad + b\,Sin.\,(A+B) + c\,Sin.\,(A+B+C)$
$\quad + d\,Sin.\,(A+B+C+D)$

* *Vide* Note in preceding page.

$$Ff = Ee + e\,Sin.(A + B + C + \ldots + E) = q + a\,Sin.\,A + b\,Sin.\,(A + B) + c\,Sin.\,(A + B + C) + d\,Sin.(A+B+C+D) + e\,Sin.(A+B+C+\ldots E)$$

$$\begin{array}{ccc} \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \end{array}$$

$$Ll = Kk + k\,Sin.(A+B+C+\ldots+K) = q + a\,Sin.\,A + b\,Sin.(A+B) + c\,Sin.(A+B+C) + d\,Sin.(A+B+C+D) + e\,Sin.\,(A+B+\ldots+E) + f\,Sin.\,(A+B+\ldots+F) + g\,Sin.\,(A+B+\ldots+G) + h\,Sin.\,(A+B+\ldots+H) + i\,Sin.\,(A+B+\ldots+I) + k\,Sin.\,(A+B+\ldots+K).$$

Likewise,

$$Ra = p$$
$$Rb = p + a\,Cos.\,A$$
$$Rc = p + a\,Cos.\,A + b\,Cos.\,(A + B)$$
$$Rd = p + a\,Cos.\,A + b\,Cos.\,(A + B) + c\,Cos.\,(A+B+C)$$

$$\begin{array}{ccc} \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \end{array}$$

$$Rl = p + a\,Cos.\,A + b\,Cos.\,(A + B) + c\,Cos.\,(A + B + C) + \ldots\ldots\ldots + k\,Cos.\,(A + B + C + \ldots\ldots + K).$$

Cor. These formulæ also obtain, when, as in *fig.* 97, the crooked line is intersected by the line of abscissæ, or when the beginning of the abscissæ is within the crooked line. For it is only necessary to suppose, that the line of abscissæ and the beginning of the abscissæ, are first without the crooked line, and that this last moves more to the right towards Q, but the former parallel with itself vertically: then, because the angles A, B, C, D, &c. and the lines of division a, b, c, d, &c. remain the same, the expressions obtained undergo no further change, than that the co-ordinates of the first point differ. If $\therefore$ these are properly determined

according to the alteration in the position, then all remains the same.

If the line of abscissæ passes through A, and we also assume this point as the origin of the abscissæ; then $p = 0$, $q = 0$, and $\therefore$, when y denotes any ordinate, and x any abscissa,

$$y = a\, Sin.\, A + b\, Sin.\, (A + B) + c\, Sin.\, (A + B + C) + \&c.$$
$$x = a\, Cos.\, A + b\, Cos.\, (A + B) + c\, Cos.\, (A + B + C) + \&c.$$

EXAM. Let (*fig.* 98) $a = 542$, $b = 698$, $c = 511$, $d = 469$, $e = 970$, $f = 902$, $g = 689$, $h = 660$, $i = 299$, $k = 783$, $p = 817$, $q = 711$; $A = -47^{\circ}.\ 45'$, $B = +94^{\circ}.\ 31'$, $C = +66^{\circ}.\ 19'$, $D = +84^{\circ}.\ 15'$, $E = +92^{\circ}.\ 34'$, $F = +126^{\circ}.\ 32'$, $G = -115^{\circ}.\ 12'$, $H = +168^{\circ}.\ 49'$, $I = +33^{\circ}.\ 5'$, $K = -161^{\circ}.\ 50'$.

Here

$Sin.\, A = Sin. - 47^{\circ}.\ 45', = - Sin.\, 47^{\circ}.\ 45',$
$Cos.\, A = Cos. - 47^{\circ}.\ 45' = Cos.\, 47^{\circ}.\ 45';$
$Sin.\, (A + B) = Sin.\, 46^{\circ}.\ 46',$
$Cos.\, (A + B) = Cos.\, 46^{\circ}.\ 46';$
$Sin.\, (A + B + C) = Sin.\, 113^{\circ}.\ 5' = Sin.\, 66^{\circ}.\ 55',$
$Cos.\, (A + B + C) = Cos.\, 113^{\circ}.\ 5' = - Cos.\, 66^{\circ}.\ 55';$
$Sin.\, (A + B + C + D) = Sin.\, 197^{\circ}.\ 20' = - Sin.\, 17^{\circ}.\ 20',$
$Cos.\, (A + B + C + D) = Cos.\, 197^{\circ}.\ 20' = - Cos.\, 17^{\circ}.\ 20';$
$Sin.\, (A + B + \ldots + E) = Sin.\, 289^{\circ}.\ 54' = - Sin.\, 70^{\circ}.\ 6',$
$Cos.\, (A + B + \ldots + E) = Cos.\, 289^{\circ}.\ 54' = Cos.\, 70^{\circ}.\ 6';$
$Sin.\, (A + B + \ldots + F) = Sin.\, 416^{\circ}.\ 26' = Sin.\, 56^{\circ}.\ 26',$
$Cos.\, (A + B + \ldots + F) = Cos.\, 416^{\circ}.\ 26' = Cos.\, 56^{\circ}.\ 26';$
$Sin.\, (A + B + \ldots + G) = Sin.\, 301^{\circ}.\ 14' = - Sin.\, 58^{\circ}.\ 46',$
$Cos.\, (A + B + \ldots + G) = Cos.\, 301^{\circ}.\ 14' = Cos.\, 58^{\circ}.\ 46';$
$Sin.\, (A + B + \ldots + H) = Sin.\, 470^{\circ}.\ 3' = Sin.\, 69^{\circ}.\ 57',$
$Cos.\, (A + B + \ldots + H) = Cos.\, 470^{\circ}.\ 3' = - Cos.\, 69^{\circ}.\ 57';$
$Sin.\, (A + B + \ldots + I) = Sin.\, 503^{\circ}.\ 8' = Sin.\, 36^{\circ}.\ 52',$
$Cos.\, (A + B + \ldots + I) = Cos.\, 503^{\circ}.\ 8' = - Cos.\, 36^{\circ}.\ 52';$

$$Sin.\,(A+B+\ldots+K) = Sin.\,341^\circ.\,18' = -\,Sin.\,18^\circ.\,42',$$
$$Cos.\,(A+B+\ldots+K) = Cos.\,341^\circ.\,18' = Cos.\,18^\circ.\,42'.$$

Therefore we have

$$\begin{array}{ll}
a\ Sin.\ A & = -542\ Sin.\ 47^\circ.45' = -401\cdot1981 \\
b\ Sin.\ (A+B) & = +698\ Sin.\ 46^\circ.46' = +508\cdot5420 \\
c\ Sin.\ (A+B+C) & = +511\ Sin.\ 66^\circ.55' = +470\cdot0870 \\
d\ Sin.\ (A+B+C+D) & = -469\ Sin.\ 17^\circ.20' = -139\cdot7292 \\
e\ Sin.\ (A+B+\ldots+E) & = -970\ Sin.\ 70^\circ.\ 6' = -912\cdot0794 \\
f\ Sin.\ (A+B+\ldots+F) & = +902\ Sin.\ 56^\circ.26' = +751\cdot5852 \\
g\ Sin.\ (A+B+\ldots+G) & = +689\ Sin.\ 58^\circ.46' = -589\cdot1382 \\
h\ Sin.\ (A+B+\ldots+H) & = +660\ Sin.\ 69^\circ.57' = +619\cdot9998 \\
i\ Sin.\ (A+B+\ldots+I) & = +299\ Sin.\ 36^\circ.52' = +179\cdot3865 \\
k\ Sin.\ (A+B+\ldots+K) & = -783\ Sin.\ 18^\circ.42' = -251\cdot0400
\end{array}$$

$$\begin{array}{ll}
a\ Cos.\ A & = +542\ Cos.\ 47^\circ.45' = +364\cdot4228 \\
b\ Cos.\ (A+B) & = +698\ Cos.\ 46^\circ.46' = +478\cdot1098 \\
c\ Cos.\ (A+B+C) & = -511\ Cos.\ 66^\circ.55' = -200\cdot3475 \\
d\ Cos.\ (A+B+C+D) & = -469\ Cos.\ 17^\circ.20' = -447\cdot7015 \\
e\ Cos.\ (A+B+\ldots+E) & = +970\ Cos.\ 70^\circ.\ 6' = +330\cdot1681 \\
f\ Cos.\ (A+B+\ldots+F) & = +902\ Cos.\ 56^\circ.26' = +498\cdot7219 \\
g\ Cos.\ (A+B+\ldots+G) & = +689\ Cos.\ 58^\circ.46' = +357\ 2633 \\
h\ Cos.\ (A+B+\ldots+H) & = -660\ Cos.\ 69^\circ.57' = -226\cdot2744 \\
i\ Cos.\ (A+B+\ldots+I) & = -299\ Cos.\ 36^\circ.52' = -239\cdot2101 \\
k\ Cos.\ (A+B+\ldots+K) & = +783\ Cos.\ 18^\circ.42' = +741\cdot6656
\end{array}$$

Hence it follows that

$$\begin{array}{lllll}
Bb = & 711 & - & 401\cdot1981 = & 309\cdot8019 \\
Cc = & Bb & + & 508\cdot5420 = & 818\cdot3439 \\
Dd = & Cc & + & 470\cdot0870 = & 1288\cdot4309 \\
Ee = & Dd & - & 139\cdot7292 = & 1148\cdot7017 \\
Ff = & Ee & - & 912\cdot0794 = & 236\cdot6223 \\
Gg = & Ff & + & 751\cdot5852 = & 988\cdot2075 \\
Hh = & Gg & - & 589\cdot1382 = & 399\cdot0693 \\
Ii = & Hh & + & 619\cdot9998 = & 1019\cdot0691 \\
Kk = & Ii & + & 179\cdot3865 = & 1198\cdot4556 \\
Ll = & Kk & - & 251\cdot0400 = & 947\cdot4156
\end{array}$$

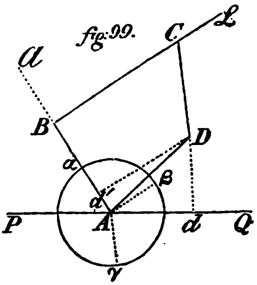
and in like manner, $Rb = 1181 \cdot 4228$, $Rc = 1659 \cdot 5326$, $Rd = 1459 \cdot 1851$, $Re = 1011 \cdot 4836$, $Rf = 1341 \cdot 6517$, $Rg = 1840 \cdot 3736$, $Rh = 2197 \cdot 6369$, $Ri = 1971 \cdot 3625$, $Rk = 1732 \cdot 1524$, $Rl = 2473 \cdot 8180$.

REMARK. If the co-ordinates of the crooked line are found, it is easy to express this line. The formulæ here found form, besides, the basis of all polygonometry, as we shall now show.

SECTION LXXXVI.

PROB. *In a quadrilateral, three sides, and the two angles between these sides, are given: find the fourth side, and the remaining angles.*

SOLUT. Let $ABCD$ (*fig.* 99) be the quadrilateral; AB, BC, CD, the given sides, and ABC, BCD, the given angles. Let $AB = a$, $BC = b$, $CD = c$.



1. Consider $ABCD$ as a crooked line, which both begins and ends in A, and through this point draw any line PQ, which may be its line of abscissæ. Let A be the beginning of the abscissæ, and about this point describe a circle; let $A\alpha$, $A\beta$, $A\gamma$, be the corresponding radii of the sides AB, BC, CD. By comparing *figs.* 97, 98, we get $A = +QA\alpha$, $B = -\alpha A\beta = -aBC$, $C = -\beta A\gamma = -bCD$.

2. Draw Dd perpendicular to PQ; then by § LXXXV when, for shortness sake, we put $A = \alpha$, $A + B = \beta$, $A + B + C = \gamma$, we obtain,

$$Dd = a\, Sin.\, \alpha + b\, Sin.\, \beta + c\, Sin.\, \gamma;$$
$$Ad = a\, Cos.\, \alpha + b\, Cos.\, \beta + c\, Cos.\, \gamma;$$

consequently

$$Dd^2 = a^2 \text{ Sin.}^2 \alpha + b^2 \text{ Sin.}^2 \beta + c^2 \text{ Sin.}^2 \gamma + 2\, ab \text{ Sin. } \alpha \text{ Sin. } \beta$$
$$+ 2\, ac \text{ Sin. } \alpha \text{ Sin. } \gamma + 2\, bc \text{ Sin. } \beta \text{ Sin. } \gamma$$
$$AD^2 = a^2 \text{ Cos.}^2 \alpha + b^2 \text{ Cos.}^2 \beta + c^2 \text{ Cos.}^2 \gamma + 2\, ab \text{ Cos. } \alpha \text{ Cos. } \beta$$
$$+ 2\, ac \text{ Cos. } \alpha \text{ Cos. } \gamma + 2\, bc \text{ Cos. } \beta \text{ Cos. } \gamma.$$

Hence, because $AD = \sqrt{(Dd^2 + Ad^2)}$, we obtain

$$AD = \sqrt{\begin{bmatrix} a^2 + b^2 + c^2 + 2\, ab \text{ Cos. } (\beta - \alpha) \\ + 2\, ac \text{ Cos. } (\gamma - \alpha) + 2\, bc \text{ Cos. } (\gamma - \beta) \end{bmatrix}}$$

(*fig.* 1 *and* 10.)

or, since $\beta - \alpha = B$, $\gamma - \alpha = B + C$, $\gamma - \beta = C$,

$$AD = \sqrt{\begin{bmatrix} a^2 + b^2 + c^2 + 2\, ab \text{ Cos. } B \\ + 2\, ac \text{ Cos. } (B + C) + 2\, bc \text{ Cos. } C \end{bmatrix}}$$

an expression, which, as must be the case, does not depend on the angle A.

3. We must further determine the two angles BAD, ADC, or even one of them only, because we then know three angles, and consequently the fourth. With this view, suppose the line PQ turns round the point A, and moves towards AD: then we have

$$Dd = a \text{ Sin. } A + b \text{ Sin. } (A + B) + c \text{ Sin. } (A + B + C) = 0,$$

where $A = DAB$.

4. By solving this equation, we obtain (*fig.* 9)

$$\begin{bmatrix} a \text{ Sin. } A + b\, [\text{Sin. } A \text{ Cos. } B + \text{Cos. } A \text{ Sin. } B] \\ + c[\text{Sin. } A \text{ Cos. } (B + C) + \text{Cos. } A \text{ Sin. } (B + C)] \end{bmatrix} = 0,$$

and by dividing by $\text{Sin. } A$,

$$\begin{bmatrix} a + b\, [\text{Cos. } B + \text{Cot. } A \text{ Sin. } B] \\ + c[\text{Cos. } (B + C) + \text{Cot. } A \text{ Sin. } (B + C)] \end{bmatrix} = 0;$$

hence further,

$$\text{Cot. } A = \text{Cot. } DAB = -\frac{a + b \text{ Cos. } B + c \text{ Cos. } (B + C)}{b \text{ Sin. } B + c \text{ Sin. } (B + C)}.$$

5. If the line PQ be so turned that it falls on AB: then we have $A = QA\alpha = 180^\circ$. On this supposition the line Dd is converted into Dd', and the line Ad into $-Ad'$. The two equations

$$Dd = a \text{ Sin. } A + b \text{ Sin. } (A + B) + c \text{ Sin. } (A + B + C)$$
$$Ad = a \text{ Cos. } A + b \text{ Cos. } (A + B) + c \text{ Cos. } (A + B + C)$$

are transformed into the following ones:

$$Dd' = a\,Sin.\,180^0 + b\,Sin.(180^0 + B) + c\,Sin.(180^0 + B + C)$$
$$-Ad' = a\,Cos.\,180^0 + b\,Cos.(180^0 + B) + c\,Cos.(180^0 + B + C)$$

or

$$Dd' = -\,b\,Sin.\,B - c\,Sin.\,(B + C)$$
$$Ad' = a + b\,Cos.\,B + c\,Cos.\,(B + C).$$

Hence we obtain, as in 4,

$$Cot.\,DAB = \frac{Ad'}{Dd'} = -\frac{a + b\,Cos.\,B + c\,Cos.\,(B + C)}{b\,Sin.\,B + c\,Sin.\,(B + C)}.$$

6. If the side AD be calculated; then also the following formula may be used to determine the angle DAB:

$$Sin.\,DAB = \frac{Dd'}{AD} = -\frac{b\,Sin.\,B + c\,Sin.\,(B + C)}{AD}.$$

Exam. Let $AB = a = 452$, $BC = b = 610$, $CD = c = 411$, $ABC = 92^0.\,5'$, $BCD = 68^0.\,53'$. Here $B = -\,aBC = -\,87^0.\,55'$, $C = -\,bCD = -\,111^0.\,7'$, and $\therefore$ $B + C = -\,199^0.\,2'$; we then have, $Cos.\,B\;Cos. -\,87^0.\,55' = Cos.\,87^0.\,55'$, $Cos.\,C = Cos. -111^0.\,7' = Cos.\,111^0.\,7' = -\,Cos.\,68^0.\,53'$, $Cos.\,(B + C) = Cos. -199^0.\,2' = Cos.\,199^0.\,2' = -\,Cos.\,19^0.\,2'$, $Sin.\,B = Sin. -87^0.\,55' = -\,Sin.\,87^0.\,55'$, $Sin.\,(B + C) = Sin. -\,199^0.\,2' = -\,Sin.\,199^0.\,2' = Sin.\,19^0.\,2'$;

$$\begin{aligned}
a^2 + b^2 + c^2 &= \quad 745325\\
2\,ab\,Cos.\,B &= \quad 20046\ 5\\
2\,ac\,Cos.\,(B + C) &= -\,351231{\cdot}3\\
2\,bc\,Cos.\,C &= -\,180645{\cdot}6\,;
\end{aligned}$$

consequently by the formula in 2,

$$AD = \sqrt{}\,(745325 + 20046\ 5 - 351231{\cdot}3 - 180645{\cdot}6)$$
$$= 483{\cdot}2127.$$

Further we have,

$$b\,Cos.\,B = 22{\cdot}1753,\quad b\,Sin.\,B = -\,609{\cdot}5966$$
$$c\,Cos.\,(B + C) = -\,388{\cdot}5302,\quad c\,Sin.\,(B + C) = 134{\cdot}0346\,;$$

consequently

$$a + b\ Cos.\ B + c\ Cos.\ (B + C) = \quad 85\ 6451$$
$$b\ Sin.\ B + c\ Sin.\ (B + C) = -475{\cdot}5620;$$

$\therefore$ by the formula in 4, 5,

$$Cot.\ DAB = \frac{85{\cdot}6451}{475{\cdot}5620} = 0{\cdot}1800924$$

and $\qquad DAB = 79^{0}.\ 47'.\ 26''.$

By the form in 6, we have

$$Sin.\ DAB = \frac{475{\cdot}5620}{483{\cdot}2127} = 0{\cdot}9841670$$

and $\qquad DAB = 79^{0}.\ 47'.\ 26''$

as before.

When the angle DAB is found, we have also ADC: thus it is

$$ADC = 360^{0} - ABC - BCD - BAD = 119^{0}.\ 14'.\ 34''.$$

SECTION LXXXVII.

Prob. *Four sides of a pentagon, and the three angles included by them, are given: find its fifth side and the two remaining angles.*

Solut. Let $ABCDE$ (*fig.* 100) be the pentagon, AB, BC, CD, DE, the given sides and ABC, BCD, CDE, the given angles. Let $AB = a$, $BC = b$, $CD = c$, $DE = d$.

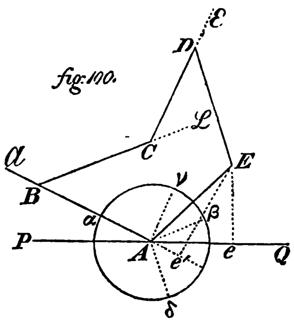


fig. 100.

1. Through A, one of the extremities of the required line AE, let any line PQ be drawn, and from the other extremity E the perpendicular Ee. Further, let a circle be described about A, and let $A\alpha$, $A\beta$, $A\gamma$, $A\delta$, according to their order be the corresponding radii of AB, BC, CD, DE.

From the opposite situation of these radii we obtain for the present figure, $A = + QA\alpha$, $B = - \alpha AB = - aBC$, $C = + \beta A\gamma = + bCD$, $D = - \gamma A \delta = - cDE$.

2. By § LXXXIV, in the crooked line $ABCDEA$, when, for shortness sake, we put $A = \alpha$, $A + B = \beta$, $A + B + C = \gamma$, $A + B + C + D = \delta$, we get

$$Ee = a \, Sin. \, \alpha + b \, Sin. \, \beta + c \, Sin. \, \gamma + d \, Sin. \, \delta$$
$$Ae = a \, Cos. \, \alpha + b \, Cos. \, \beta + c \, Cos. \, \gamma + d \, Cos. \, \delta,$$

and $\therefore$

$$\begin{aligned} Ee^2 = {} & a^2 \, Sin.^2 \, \alpha + b^2 \, Sin.^2 \, \beta + c^2 \, Sin.^2 \, \gamma + d^2 \, Sin.^2 \, \delta \\ & + 2ab \, Sin. \, \alpha \, Sin. \, \beta + 2ac \, Sin. \, \alpha \, Sin. \gamma + 2ad Sin. \, \alpha \, Sin. \, \delta \\ & + 2bc \, Sin. \, \beta \, Sin. \, \gamma + 2bd \, Sin. \, \beta \, Sin. \, \delta + 2cd \, Sin. \, \gamma \, Sin. \delta \end{aligned}$$

$$\begin{aligned} Ae^2 = {} & a^2 \, Cos.^2 \, \alpha + b^2 \, Cos.^2 \, \beta + c^2 \, Cos.^2 \, \gamma + d^2 \, Cos.^2 \, \delta \\ & + 2ab \, Cos. \, \alpha \, Cos. \, \beta + 2ac \, Cos. \, \alpha \, Cos. \, \gamma + 2ad \, Cos. \, \alpha \, Cos. \delta \\ & + 2bc \, Cos. \, \beta \, Cos. \, \gamma + 2bd \, Cos. \beta \, Cos. \, \delta + 2cd Cos. \gamma \, Cos. \delta. \end{aligned}$$

3. Now since $AE = \sqrt{(Ee^2 + Ae^2)}$: $\therefore$ (*fig.* 1 *and* 10),

$$AE = \sqrt{\left\{ \begin{array}{c} a^2 + b^2 + c^2 + d^2 + 2 \, ab \, Cos. \, (\beta - \alpha) + \\ 2ac \, Cos. (\gamma - \alpha) + 2ad \, Cos. \, (\delta - \alpha) + 2bc Cos. \, (\gamma - \beta) \\ + \, 2bd \, Cos. \, (\delta - \beta) + 2cd \, Cos. \, (\delta - \gamma) \end{array} \right\}}$$

or, since $\beta - \alpha = B$, $\gamma - \alpha = B + C$, $\delta - \alpha = B + C + D$, $\gamma - \beta = C$, $\delta - \beta = C + D$, $\delta - \gamma = D$,

$$AE = \sqrt{\left\{ \begin{array}{c} a^2 + b^2 + c^2 + 2 \, ab \, Cos. \, B + 2 \, ac \, Cos. \, (B + C) \\ + \, 2 \, ad \, Cos. \, (B + C + D) + 2 \, bc \, Cos. \, C + \\ 2 \, bd \, Cos. \, (C + D) + 2 \, cd \, Cos. \, D \end{array} \right\}}$$

4. Let the line PQ so move towards P, that it coincides with AB: then $A = 180^0$, and Ee is converted into Ee'. We consequently have

$$\begin{aligned} Ee' = {} & a Sin. \, 180^0 + b Sin. (180 + B) + c Sin. (180^0 + B + C) \\ & + d \, Sin. \, (180^0 + B + C + D), \end{aligned}$$

or

$$Ee' = - \, b \, Sin. \, B - c \, Sin. \, (B + C) - d \, Sin. \, (B + C + D).$$

Now since $Sin. \, BAE = Sin. \, EAe' = \dfrac{Ee'}{AE}$: therefore

$$Sin.\ BAE + \frac{b\ Sin. B + c\ Sin. (B+C) + d\ Sin. (B+C+D)}{AE}.$$

If ∴ the side AE is calculated by means of the formula in 3, we then obtain from this formula the triangle BAE.

5. But if the line PQ move towards Q, in the direction AE; then $A = BAE$, and $Ee = 0$, and we have the equation,

$$\left\{\begin{array}{l} a\ Sin. A + b\ Sin. (A + B) + c\ Sin. (A + B + C) \\ + d\ Sin. (A + B + C + D) \end{array}\right\} = 0,$$

or

$$\left\{\begin{array}{l} a\ Sin.\ A + b\ [Sin.\ A\ Cos.\ B + Cos.\ A\ Sin.\ B] + \\ c\ [Sin. A\ Cos. (B + C) + Cos. A\ Sin. (B + C)] + \\ d[Sin. A\ Cos. (B + C + D) + Cos. A\ Sin. (B + C + D)] \end{array}\right\} = 0.$$

Divide this equation by $Sin.\ A$, substitute $Cot.\ A$ for $\frac{Cos. A}{Sin. A}$, and subtract $Cot.\ A = Cot.\ BAE$: this gives

$$Cot.\ BAE = -\frac{a + b Cos. B + c Cos. (B+C) + d Cos. (B+C+D)}{b\ Sin.\ B + c\ Sin.\ (B+C) + d\ Sin. (B+C+D)}$$

or also

$$Tan.\ BAE = -\frac{b\ Sin. B + c\ Sin. (B+C) + d\ Sin. (B+C+D)}{a + b Cos. B + c Cos. (B+C) + d Cos. (A+C+D)}$$

We have ∴ two different formulæ for the angle BAE, which mutually serve to prove the calculation.

Exam. Let $a = 540$, $b = 519$, $c = 438$, $d = 536$, $ABC = 46^\circ. 38'$, $BCD = 136^\circ. 5'$, $CDE = 38^\circ. 51'$; consequently $B = -133^\circ. 22'$, $C = 43^\circ. 55'$, $D = -141^\circ. 9'$. Here $Sin.\ B = Sin. - 133^\circ. 22' = - Sin.\ 133^\circ. 22' = - Sin.\ 46^\circ. 38'$, $Cos.\ B = Cos. - 133^\circ. 22' = Cos.\ 133^\circ. 22' = - Cos. 46^\circ. 38'$, $Cos.\ C = Cos.\ 43^\circ. 55'$, $Cos.\ D = Cos. - 141^\circ. 9' = Cos.\ 141^\circ.\ 9'$, $= - Cos.\ 38^\circ.\ 51$, $Sin.\ (B + C) = Sin. - 89^\circ. 27' = - Sin.\ 89^\circ. 27'$, $Sin.\ (B + C + D) = Cos. - 89^\circ. 27' = - Cos.\ 89^\circ. 27'$, $Sin.\ (B + C + D) =$

$Sin. - 230°. 36' = - Sin. 230°. 36' = Sin. 50°. 36'$, $Cos. (B+C+D) = Cos. - 230°. 36' = Cos. 230°. 36' = - Cos. 50°. 36'$, $Cos. (C+D) = Cos. - 97°. 14' = Cos\ 97°. 14' = - Cos. 82°. 46'$; consequently

$$
\begin{aligned}
a^2 + b^2 + c^2 + d^2 &= 1040101 \\
2\,ab\ Cos.\ B &= -\ 384889{\cdot}4 \\
2\,ac\ Cos.(B + C) &= 4540{\cdot}8 \\
2\,ad\ Cos.(B+C+D) &= -\ 367432{\cdot}8 \\
2\,bc\ Cos.\ C &= 327502{\cdot}5 \\
2\,bd\ Cos.\ (C+D) &= -\ 70052\ 5 \\
2\,cd\ Cos.\ D &== 365670{\cdot}3;
\end{aligned}
$$

and $\therefore$ $AE = \sqrt{184099{\cdot}3} = 429{\cdot}0679.$

Further

$$
\begin{aligned}
b\ Sin.\ B &= -\ 377{\cdot}2996 \\
b\ Cos.\ B &= -\ 356{\cdot}3720 \\
c\ Sin.\ (B + C) &= -\ 437{\cdot}9797 \\
c\ Cos.\ (B + C) &= 4{\cdot}2044 \\
d\ Sin.\ (B+C+D) &= 414{\cdot}1852 \\
d\ Cos.\ (B+C+D) &= -\ 340{\cdot}2155,
\end{aligned}
$$

$\therefore$ by the formula in 4,

$$Sin.\ BAE = \frac{401{\cdot}0941}{429{\cdot}0679} = 0{\cdot}9348033$$

and by the formula in 5,

$$Cot.\ BAE = -\ \frac{152{\cdot}3901}{401{\cdot}0941} = -\ 0{\cdot}3799360.$$

From both we obtain

$$BAE = 110°.\ 48'.\ 13''.$$

The negative co-tangent which is obtained from the second formula, shows that the angle is obtuse, which leaves the first indeterminate, because two angles belong to each sine.

When BAE is found, we have also the angle AED.

SECTION LXXXVIII.

PROB. *All the sides of a polygon except one are given, also all the angles included by these sides: find the unknown sides and the two remaining angles.*

SOLUT. The treatment of the quadrilateral and pentagon in §§ LXXXVI, LXXXVII, clearly shows the method to be adopted for every other polygon. The law by which the formulæ are governed is simple, and easily discovered: it may be expressed by words in the following way:

1. In order to find the unknown sides of the polygon, take the squares of all the given sides; take, further, twice the products of every two of these sides combined in every possible way, and multiply each of them by the cosine of the algebraical sum of the exterior angles between the respective sides; then add all together, and extract the square root from the product.

2. In order to find one of the two required angles of the polygon, assume the given line which is adjacent to the required angle for the first, and multiply each of the remaining given sides both by the sine and the cosine of the algebraical sum of all the exterior angles between it and the above first side; then add all the products arising from the cosines to the first side, and divide the sum by the sum of all the products arising from the sines; the quotient, with a different sign, gives the co-tangent of the required angle. Or, divide the sum of the products arising from the sines, by the side found by the first rule: then the quotient, with a different sign, gives the sine of the required angle.

Thus, if a, b, c, d, e, in the order in which they are here placed, be the five given sides of a hexagon, and B, C, D, E the four given exterior angles formed by these sides: then the unknown side =

$$\sqrt{\left\{\begin{array}{c} a^2+b^2+c^2+d^2+e^2+2ab\,\text{Cos.}\,B+2ac\,\text{Cos.}(B+C) \\ +2ad\,\text{Cos.}(B+C+D)+2ae\,\text{Cos.}(B+C+D+E) \\ +2bc\,\text{Cos.}\,C+2bd\,\text{Cos.}(C+D)+2be\,\text{Cos.}(C+D+E) \\ +2cd\,\text{Cos.}\,D+2ce\,\text{Cos.}(D+E)+2de\,\text{Cos.}\,E \end{array}\right\}}$$

and the cotangent of the required angle adjacent to the side $a =$

$$\frac{\left[\begin{array}{c}a + b\,Cos.\,B + c\,Cos.\,(B + C) + d\,Cos.\,(B + C + D) + \\ e\,Cos.\,(B + C + D + E)\end{array}\right]}{\left[\begin{array}{c}b\,Sin.\,B + c\,Sin.\,(B + C) + d\,Sin.\,(B + C + D) + \\ e\,Sin.\,(B + C + D + E)\end{array}\right]}$$

or the sine of this angle =

$$\frac{\left[\begin{array}{c}b\,Sin.\,B + c\,Sin.\,(B + C) + d\,Sin.\,(B + C + D) + \\ e\,Sin.\,(B + C + D + E)\end{array}\right]}{x}$$

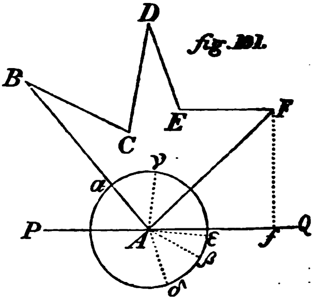
where x denotes the required side.

That these formulæ are also applicable to the triangle, and that hence the known trigonometrical rules may be derived, are necessary consequences: this subject is treated in Däzel's Elements of Goniometry; Munich, 1800.

SECTION LXXXIX.

PROB. *In a polygon, all the sides except two are given, likewise all the angles: find the two unknown sides.*

SOLUT. Thus, let $ABCDEF$ (*fig.* 101) be a hexagon, in which all the angles, and all the sides, except the two AF, CD, are known.



1. Through A draw any line of abscissæ PQ, and Ff perpendicular to it: then, when $ABCDEF$ is considered as a crooked line, by § LXXXV,

$$Ff = AB.Sin.A + BC.Sin.(A + B) + CD.Sin.(A + B + C) + DE.Sin.(A + B + C + D) + EF.Sin.(A + B + C + D + E)$$

$$Af = AB.Cos.A + BC.Cos.(A + B) + CD.Cos.(A + B + C) + DE.Cos.(A + B + C + D) + EF.Cos.(A + B + C + D + E)$$

in which $A = QAB$.

2. Let the line PQ move towards Q, so that it may fall on AF; then $Ff=0$, and $Af=AF$, and instead of the two foregoing equations, we have the following one:

$$0=AB.\ Sin.\ A+BC.\ Sin.\ (A+B)+CD.\ Sin.\ (A+B+C) +DE.Sin.(A+B+C+D)+EF.Sin.(A+B+C+D+E)$$

$$AB=AB.\ Cos.\ A+BC.\ Cos.(A+B)+CD.\ Cos.(A+B+C) +DE.Cos.(A+B+C+D)+EF.Cos.(A+B+C+D+E),$$

in which $A=FAB$.

3. From the first of these two equations, we obtain

$$CD=\frac{-\left[\begin{array}{c}AB.Sin.A+BC.Sin.(A+B)+DE.Sin.(A+B+C+D)\\ +EF.\ Sin.\ (A+B+C+D+E)\end{array}\right]}{Sin.\ (A+B+C)}$$

Having from hence determined CD; then the second equation gives the line AF.

Exam. Let $AB=1040$, $BC=624$, $DE=533$, $EF=481$; $ABC=23^\circ.\ 52'$, $BCD=69^\circ.\ 14'$, $CDE=30^\circ.\ 24'$, $DEF=115^\circ.\ 30'$, $AFE=46^\circ.\ 44'$, and consequently $BAF=83^\circ.\ 44'$. Let $A\alpha$, $A\beta$, $A\gamma$, $A\delta$, $A\varepsilon$, according to their order, be the corresponding radii of the sides AB, BC, CD, DE, EF: then it follows from the situation of these lines with respect to each other, and to AF, into which PQ moves, that $A=FA\alpha=83^\circ.\ 44'$, $B=-\alpha A\beta=-156^\circ.\ 8'$, $C=\beta A\gamma=110^\circ.46'$, $D=-\gamma A\delta=-149^\circ.\ 36'$, $E=\delta A\varepsilon=64^\circ.\ 30'$. We consequently have $Sin.\ (A+B)=Sin.-72^\circ.\ 24'=-Sin.\ 72^\circ.\ 24'$, $Cos.\ (A+B)=Cos.-72^\circ.\ 24'=Cos.\ 72^\circ.\ 24'$, $Sin.\ (A+B+C)=Sin.\ 38^\circ.\ 22'$, $Cos.(A+B+C)=Cos.\ 38^\circ.\ 22'$, $Sin.\ (A+B+C+D)=Sin.-111^\circ.\ 14'=-Sin.\ 68^\circ.\ 46'$, $Cos.\ (A+B+C+D)=Cos.-111^\circ.\ 14'=-Cos.\ 68^\circ.\ 46'$, $Sin.\ (A+B+\ldots+E)=Sin.-46^\circ.\ 44'=-Sin.\ 46^\circ.\ 44'$, $Cos.\ (A+B+\ldots+E)=Cos.-46^\circ.\ 44'=Cos.\ 46^\circ.\ 44'$; consequently,

$$
\begin{array}{lcr}
AB.\ Sin.\ A & = & 1033{\cdot}7853 \\
AB.\ Cos.\ A & = & 113{\cdot}5222 \\
BC.\ Sin.\ (A+B) & = - & 594{\cdot}7909 \\
BC.\ Cos.\ (A+B) & = & 188\ 6788 \\
DE.\ Sin.\ (A+B+C+D) & = - & 496{\cdot}8163 \\
DE.\ Cos.\ (A+B+C+D) & = - & 193{\cdot}0349 \\
EF.\ Sin.\ (A+B+C+D+E) & = - & 350{\cdot}2507 \\
EF.\ Cos.\ (A+B+C+D+E) & = & 329{\cdot}6749
\end{array}
$$

Hence we find,

$$CD = \frac{-(1033{\cdot}7853 - 594{\cdot}7909 - 496\ 8163 - 350{\cdot}2507)}{Sin.\ 38^0.\ 22'}$$

$$= \frac{408{\cdot}0726}{Sin.\ 38^0.\ 22'} = 657{\cdot}4481,$$

$$\therefore \quad CD.\ Cos.\ (A+B+C) = 515{\cdot}4753,$$

$$AF = 113{\cdot}5222 + 188{\cdot}6788 + 515{\cdot}4753 - 193{\cdot}0349 + 329{\cdot}6749 = 954{\cdot}3163.$$

In the values found for CD, AF, the error which occurs on account of the incompleteness of the logarithmic tables, cannot amount to 0·001.

REMARK. Although in the solution, the calculation, for the sake of perspicuity, has only been performed for one figure; yet it is sufficiently evident from hence, how we are to proceed in the case of every other figure.

COR. If CD be parallel to AF: then $FA\gamma = FA\alpha - \alpha A\beta + \beta A\gamma = A + B + C = 0$; consequently $Sin.\ (A + B + C) = 0$; the expression for CD, and consequently also that for AF, is determined. Of the accuracy of this result we can easily convince ourselves merely by inspecting the figure. For let the points A, B, C, retain their places, but move the angle DEF between the two parallel lines AF, CD, forwards or backwards, in such a way, that the lines DE, EF, continue to be parallel to themselves, consequently by these means neither the lines DE, EF, nor the angles CDE, DEF, AFE, undergo any change with respect to their magnitude or position; whence follows the indeterminateness of the lines CD, AF.

SECTION XC.

Prob. *In a polygon all the sides but one, and all the angles but two, are given, likewise the two unknown angles are at one of the given sides: find these angles and the unknown side.*

Solut. For the sake of perspicuity, take the pentagon $ABCDE$ (*fig.* 100); let CD be the unknown side, and BAE, DEA the unknown angles. Let $AB = a$, $BC = b$, $CD = x$, $DE = d$, $EA = e$.

1. By §§ LXXXVII, LXXXVIII, we have

$$AE^2\ (= e^2) = a^2 + b^2 + x^2 + b^2 + 2\,ab\ Cos.\ B + 2ax\ Cos.\ (B + C) + 2ad\,Cos(B + C + D) + 2bx\ Cos.\ C + 2bd\ Cos.\ (C + D) + 2dx\ Cos.\ D.$$

This equation, when solved, gives

$$x = -\,a\ Cos.\ (B + C) - b\ Cos.\ C - d\ Cos.\ D \pm \sqrt{\begin{bmatrix} e^2 - a^2 - b^2 - d^2 - 2ab\ Cos.\ B - 2ad\ Cos.\ (B + C + D) \\ -2bdCos.(C + D) + [aCos.(B + C) + bCos.C + dCos.D]^2 \end{bmatrix}}$$

Having from this determined the line CD; then the angles BAE, DEA, may be found by the formulæ given above.

Exam. Let $a = 540$, $b = 519$, $d = 536$, $e = 429{\cdot}0679$, $\beta = -\,133^0.\ 22'$, $C = 43^0.\ 55'$, $D = -\,141^0.\ 9'$; which values are taken from the example in § LXXXVII, where $AE = e$ was sought, whereas here, on the contrary, e is assumed to be known, while in the above example c was assumed to be unknown. Here $e^2 - a^2 - b^2 - d^2 = -\,664157{\cdot}73718959$; further, we have

$$\begin{aligned} a\ Cos.\ (B + C) &= -\ 540\ Cos.\ 89^0.\ 27' = \quad 5{\cdot}1835 \\ b\ Cos.\ C &= \quad 519\ Cos.\ 43^0.\ 55' = \quad 373{\cdot}8614 \\ d\ Cos.\ D &= -\ 536\ Cos.\ 38^0.\ 51' = -\ 417{\cdot}4317. \end{aligned}$$

The other members, of which the expression found for x is composed, are already calculated above. We consequently have

$$x = -5{\cdot}1835 - 373{\cdot}8614 + 417{\cdot}4317 \pm$$

$$\sqrt{\begin{bmatrix} -664157{\cdot}73718959 + 384889{\cdot}4 + 367432{\cdot}8 \\ + 70052{\cdot}5 + [5{\cdot}1835 + 373{\cdot}8614 - 417{\cdot}4317]^2 \end{bmatrix}}$$

or $\qquad x = 38{\cdot}3868 \pm 399{\cdot}6129 \ldots = 437{\cdot}9997 \ldots$

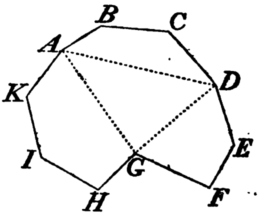
The negative value cannot be used here. In § LXXXVII, we assumed $c = 438$; the difference does not quite amount to 0·0003.

SECTION XCI.

PROB. *In a polygon all the sides but one are given, also all the angles except two, but these last are not, as in the foregoing problem, assumed to be at one side: find the unknown side, and the two unknown angles.*

SOLUT. In the polygon *ABCDEFGHIK (fig.* 102*)* all the angles, except *KAB*, *HGF*, and all the sides except *CD*, are given.

fig. 102.



Draw the diagonal *AG*, and by these means divide the polygon into two others. In the polygon *AKIHG*, all the sides, except *AG*, and the angles *K*, *I*, *H*, included by the given sides, are given; consequently by § LXXXVIII, the two unknown angles *KAG*, *AGH*, and the side *AG* may be found. If *AG* be found, then in the polygon *ABCDEFG*, we have all the sides except *CD*, and all the angles except *BAG*, *AGF*; these may ∴ be found by § XC. Having by these means determined the angles *KAG*, *AGH*, *BAG*, *AGF*; we consequently have $KAB = KAG + BAG$, $HGF = 360^\circ - AGH - AGF$.

SECTION XCII.

Prob. *In a polygon, all the sides, and all the angles except three, are given: find the unknown angles.*

Solut. Let $ABC \ldots K$ (*fig.* 102) be a polygon, all of whose sides, and all its angles except the three A, D, G, are given.

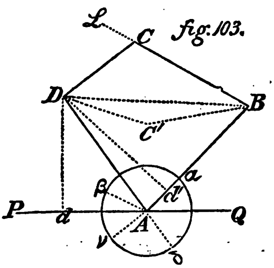
Draw the diagonals AD, AG, DG. Then in the polygon $ABCD$ all the sides except AD, and the angles included by these sides, are known; consequently (§ LXXXVIII) the angles BAD, CDA, together with the side AD, may be found. In like manner, in the polygon $DEFG$, we find the side DG, and the angles EDG, FGD, and in the polygon $AKIHG$ the side AG, and the angles KAG, HGA. Since $\therefore$ the three sides of the triangle ADG are known; consequently we also have its angles, from which, and the above-named, the angles A, D, G, may be determined. Thus we have $KAB = KAG + GAD + DAB$, $CDE = CDA + ADG + GDE$, $HGF = 360^0 - (HGA + AGD + DGF)$.

SECTION XCIII.

Prob. *In a quadrilateral, three sides, and the angles included by them, are given: find its area.*

Solut. Let $ABCD$ (*fig.* 103) be the quadrilateral whose area is sought; $AB = a$, $BC = b$, $CD = c$, the given sides, and ABC, BCD, the given angles. Let PQ be any line of abscissa, Dd perpendicular to it, and $A\alpha$, $A\beta$, $A\gamma$, $A\delta$, the corresponding radii of the sides AB, BC, CD, DA.

fig. 103.



1. By § XXVI, $\triangle\, BCD = \frac{1}{2} bc\, Sin.\, BCD$; or since $BCD = 180^0 - bCD = 180^0 - \beta A\gamma = 180^0 - C$,

$$\triangle\, BCD = \tfrac{1}{2} bc\, Sin.\, C$$

2. By § LXXXV we have
$Dd = a\,Sin.\,A + b\,Sin.\,(A + B) + c\,Sin.\,(A + B + C)$, where $A = QA\kappa$.

3. Let the line PQ so move, that it coincides with AB: then $A = 0$, and Dd is transformed into Dd'; we consequently have

$$Dd' = b\,Sin.\,B + c\,Sin.\,(B + C),$$

and $\therefore$

$$\triangle BAD = \tfrac{1}{2}\,AB \times Dd' = \tfrac{1}{2}\,ab\,Sin.\,B + \tfrac{1}{2}\,ac\,Sin.\,(B + C).$$

4. Now from 1 and 3 it follows, that $ABCD = \triangle\,BAD + \triangle BCD = \tfrac{1}{2}\,[ab Sin. B + ac Sin. (B + C) + bc Sin.\,C]$. This is the same expression, which, by a suitable alteration of the exterior angles into interior, was found in § XXXVII, by another method.

Cor. The expression found is still correct, when the quadrilateral has an angle tending inwards. For let the point C be in C': we must then take the triangle BCD away from the triangle BAD, instead of adding it as before.

But for this case also, the angle C is negative, which follows from the change which takes place in the position of the corresponding radii, and we obtain for the triangle $BCD = \tfrac{1}{2}\,bc\,Sin.\,C$ a negative value.

Exam. Let $A = 682$, $b = 616$, $e = 407$, $B = 113°. 46'$, $C = 63°. 49'$. Here $ab\,Sin.\,B = 384484{\cdot}1$; $ac\,Sin.\,(B + C) = 11704{\cdot}3$; $bc\,Sin.\,C = 224985{\cdot}6$; consequently the quadrilateral $ABCD = 310587$.

SECTION XCIV.

Prob. *In a pentagon, four sides, and the angles included by them, are given: find its area.*

Solut. Let $ABCDE$ (*fig.* 104) be the pentagon;

$AB = a$, $BC = b$, $CD = c$, $DE = d$, the given sides, and B, C, D, the given angles.

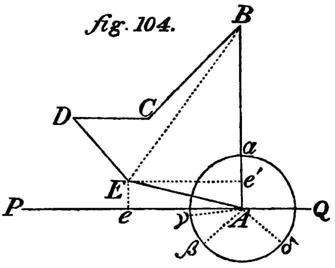


fig. 104.

1. Divide the pentagon by the diagonal BE, into the quadrilateral $BCDE$, and the triangle BAE; through A draw any line of abscissæ PQ, and the perpendicular Ee.

2. Then by § LXXXV,

$$Ee = a \text{ Sin. } A + b \text{ Sin. } (A + B) + c \text{ Sin. } (A + B + C) + d \text{ Sin. } (A + B + C + D),$$

in which $A = QA\alpha$. Now let the line PQ move towards Q, so that it coincides with AB: then $A = 0$, and Ee is transformed into Ee'; we consequently have

$$Ee' = b \text{ Sin. } B + c \text{ Sin. } (B + C) + d \text{ Sin. } (B + C + D)$$

and $\therefore \triangle BEA = \frac{1}{2} AB \times Ee' =$

$$\tfrac{1}{2}[ab \text{ Sin. } B + ac \text{ Sin. } (B + C) + ad \text{ Sin. } (B + C + D)].$$

3. But by the foregoing section, quadrilateral $BCDE = \frac{1}{2}[bc \text{ Sin. } C + bd \text{ Sin. } (C + D) + cd \text{ Sin. } D]$; consequently pentagon $ABCDE =$

$$\tfrac{1}{2}\left[\begin{array}{l} ab \text{ Sin. } B + ac \text{ Sin. } (B + C) + ad \text{ Sin. } (B + C + D) \\ + bc \text{ Sin. } C + bd \text{ Sin. } (C + D) + cd \text{ Sin. } D \end{array}\right]$$

Exam. Let $a = 332$, $b = 248$, $c = 128$, $d = 152$, $B = 132^\circ.\ 14'$, $C = -38^\circ.\ 29'$, $D = 140^\circ.\ 47'$. Here $\text{Sin. } B = \text{Sin. } 47^\circ.\ 46'$, $\text{Sin. } (B + C) = \text{Sin. } 93^\circ.\ 45' = \text{Sin. } 86^\circ.\ 15'$, $\text{Sin. } (B + C + D) = \text{Sin. } 234^\circ.\ 32' = -\text{Sin. } 54^\circ.\ 32'$, $\text{Sin. } C = -\text{Sin. } 38^\circ.\ 29'$, $\text{Sin. } (C + D) = \text{Sin. } 102^\circ.\ 18' = \text{Sin. } 77^\circ.\ 42'$, $\text{Sin. } D = \text{Sin. } 39^\circ.\ 13'$; we consequently have

$$ab \text{ Sin. } B = 60962{\cdot}69$$
$$ac \text{ Sin. } (B + C) = 42405{\cdot}01$$

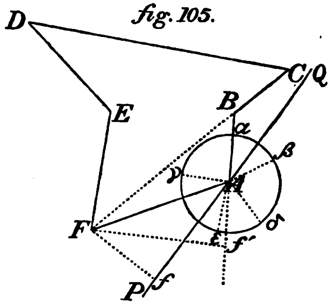
$$\begin{aligned} ad \; Sin. \; (B + C + D) &= - \; 41100{\cdot}56 \\ bc \; Sin. \; C &= - \; 19753{\cdot}88 \\ bd \; Sin. \; (C + D) &= \quad 36830{\cdot}71 \\ cd \; Sin. \; D &= \quad 12301{\cdot}15 \, ; \end{aligned}$$

and $\therefore$ pentagon $ABCDE = 45822{\cdot}56$.

SECTION XCV.

PROB. *In a hexagon, five sides, and the four angles included by them, are given : find its area.*

SOLUT. Let $ABCDEF$ (*fig.* 105) be the hexagon; $AB = a$, $BC = b$, $CD = c$, $DE = d$, $EF = e$, the given sides, and B, C, D, E, the given angles.



1. Draw the diagonal BF, and by it divide the hexagon into the pentagon $BCDEF$, and the triangle BAF; further, let PQ be the arbitrary line of abscissæ, and Ff, perpendicular to it : then (§ LXXXV),

$$Ff = a \; Sin. \; A + b \; Sin. \; (A + B) + c \; Sin. \; (A + B + C) + d \; Sin. \; (A+B+C+D) + e \; Sin. \; (A+B+C+D+E),$$

in which $A = QA\alpha$. Let the line PQ move into AB: then $A = 0$, and Ff is transformed into Ff'; we consequently have

$$Ff' = b \; Sin. B + c \; Sin. (B + C) + d \; Sin. (B + C + D) + e \; Sin. \; (B + C + D + E),$$

and $\therefore$ $\quad \triangle BAF = \frac{1}{2} AB \times Ff' =$

$$\frac{1}{2}\left\{\begin{matrix} ab \; Sin. \; B + ac \; Sin. \; (B \parallel C) + ad \; Sin. \; (B+C+D) \\ + ae \; Sin. \; (B+C+D+E) \end{matrix}\right\}$$

2. By the preceding section, pentagon $BCDEF =$

$$\frac{1}{2}\left\{\begin{matrix} bc \; Sin. \; C + bd \; Sin. \; (C+D) + be \; Sin. \; (C+D+E) \\ + cd \; Sin. \; D + ce \; Sin. \; (D+E) + de \; Sin. \; E \end{matrix}\right\}$$

we $\therefore$ have hexagon $ABCDEF =$

$$\tfrac{1}{2}\left\{\begin{array}{c} ab\ Sin.\ B + ac\ Sin.\ (B+C) + ad\ Sin.\ (B+C+D) \\ + ae Sin.(B+C+D+E) + bc\ Sin. C + bd Sin.(C+D) \\ + be\ Sin.\ (C+D+E) + cd\ Sin.\ D + ce\ Sin.\ (D+E) \\ + de\ Sin.\ E \end{array}\right\}$$

Exam. Let $a = 324$, $b = 288$, $c = 1102$, $d = 528$, $e = 504$, $B = -\alpha A\beta = -60^\circ.\ 52'$, $C = \beta A\gamma = 129^\circ.\ 20'$, $D = \gamma A\delta = 139^\circ.\ 43'$, $E = \delta A\varepsilon = -55^\circ.\ 50'$. Here

$ab\ Sin.\ B$	$=$	$-\ 81507\cdot02$
$ac\ Sin.\ (B+C)$	$=$	$332127\cdot56$
$ad\ Sin.\ (B+C+D)$	$=$	$-\ 80796\cdot33$
$ae\ Sin.\ (B+C+D+E)$	$=$	$85780\cdot64$
$bc\ Sin.\ C$	$=$	$245481\cdot35$
$bd\ Sin.\ (C+D)$	$=$	$-\ 152043\cdot07$
$be\ Sin.\ (C+D+E)$	$=$	$-\ 79515\cdot22$
$cd\ Sin.\ D$	$=$	$376209\cdot31$
$ce\ Sin.\ (D+E)$	$=$	$552245\cdot95$
$de\ Sin.\ E$	$=$	$-\ 229182\cdot98$;
and $\therefore$ hexagon $ABCDEF$	$=$	$483900\cdot09$.

SECTION XCVI.

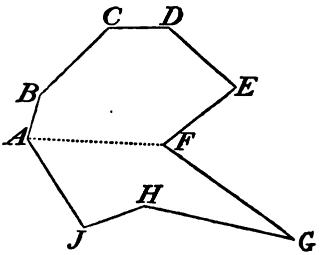
Prob. *All the sides of a polygon but one, and all the angles included by these sides, are given: find its area.*

Solut. From §§ XCIII, XCIV, XCV, the law may be easily perceived, by which the area of every polygon may be calculated from the parts given; this may be expressed in words as follow:

To calculate the area of a polygon from the parts given, assume half the sum of the product of every two of the given sides, combined in every possible way, each multiplied by the sine of the algebraical sum of the exterior angles lying between them.

Cor. If a polygon has a great number of sides, as *ABCDEFGHI (fig.* 106*)*, and if all its sides and angles are known: then the best plan would be, to divide it by a diagonal *AE* into two other polygons *ABCDEF*, *FGHIA*, so that one may have the same number of sides as the other, or one more, and then to calculate each separately, while we consider *AF* as the side of both polygons which is not given. By these means we obtain this advantage, that it is not necessary to calculate so many terms as we otherwise should have: likewise the sums of the angles are also less, by which the calculation is essentially shortened.

fig. 106.



SECTION XCVII.

Prob. *All the angles and sides of a polygon are given: find any arbitrary diagonal of this figure.*

Solut. Let *ABCDEFGHI* (*fig.* 106) be any polygon, and *AF* the diagonal sought.

In the polygon *ABCDEF*, which is divided by the diagonal *AF*, the sides *AB*, *BC*, *CD*, *DE*, *EF*, and the angles included by them, viz. *B*, *C*, *D*, *E*, are given: it is required ∴ to find *AF* (§ LXXXVIII).

Cor. But the line *AF* may also be found from the other polygon *FGHIA*, which is cut off by this diagonal. We consequently have two different expressions for these lines; and since this likewise obtains for every other diagonal; hence ∴ there are a great number of equations, which express the relations between the sides and angles of a polygon. Thus, for the diagonal *AF*, when the sides *AB*, *BC*, *CD*, *DE*, *EF*, *FG*, *GH*, *HI*, *IA*, are respectively denoted by *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, we obtain the following equation:

$$a^2 + b^2 + c^2 + d^2 + e^2 + 2\,ab\,\text{Cos.}\,B + 2\,ac\,\text{Cos.}\,(B + C) + 2\,ad\,\text{Cos.}(B + C + D) + 2\,ae\,\text{Cos.}(B + C + D + E) + 2\,bc\,\text{Cos.}\,C + 2\,bd\,\text{Cos.}(C + D) + 2be\text{Cos.}(C + D + E) + 2\,cd\,\text{Cos.}\,D + 2\,ce\,\text{Cos.}\,(D + E) + 2\,de\,\text{Cos.}\,E = f^2 + g^2 + h^2 + i^2 + 2\,fg\,\text{Cos.}\,G + 2\,fh\,\text{Cos.}\,(G + H) + fi\,\text{Cos.}(G + H + I) + 2\,gh\,\text{Cos.}\,H + 2\,gi\,\text{Cos.}(H + I) + 2\,hi\,\text{Cos.}\,I.$$

SECTION XCVIII.

Prob. *To determine the distance between two places, when it is rather considerable.*

Solut. Let A, F, (*fig.* 107) be the two places, whose distance is required.

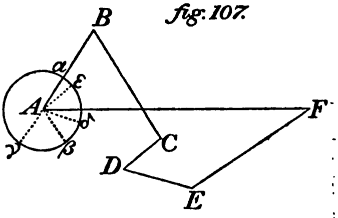


fig: 107.

Connect A, F, with proper distances AB, BC, CD, DE, EF: measure these lines in a direct or indirect way, also the angles included by them, viz. ABC, BCD, CDE, DEF: hence AF may be determined by § LXXXVIII. Thus, if we denote, as before, the sides AB, BC, CD, DE, EF, by a, b, c, d, e, respectively, and the exterior angles by B, C, D, E: then

$$AF = \sqrt{\begin{bmatrix} a^2 + b^2 + c^2 + d^2 + e^2 + 2\,ab\,\text{Cos.}\,B + \\ 2\,ac\,\text{Cos.}(B + C) + 2\,ad\,\text{Cos.}(B + C + D) + \\ 2\,ae\,\text{Cos.}(B + C + D + E) + 2\,bc\,\text{Cos.}\,C + \\ 2\,bd\,\text{Cos.}(C + D) + 2\,be\,\text{Cos.}(C + D + E) + \\ 2\,cd\,\text{Cos.}\,D + 2\,ce\,\text{Cos.}\,(D + E) + 2\,de\,\text{Cos.}\,E \end{bmatrix}}$$

where $B = -\alpha A\beta = -(180^0 - ABC)$, $C = -\beta A\gamma = -(180^0 - BCD)$, $D = \gamma A\delta = 180^0 - CDE$, $E = \delta A\varepsilon = 180^0 - DEF$.

Remark. The extension which has been here given to the problem in § LXXXVIII, is correct, if in the proofs on which its solution is founded, nothing is assumed which is not applicable to every divided line included by it.

SECTION XCIX.

Prob. *To find the area of a polygon from one of its sides, the two angles adjacent to it, and the angles, which the diagonals drawn from the angular points of this side make with it.*

Solut. Let $ABCDEFG$ (*figs.* 108, 109) be a polygon,

fig. 108.

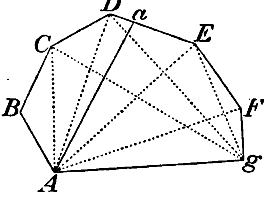
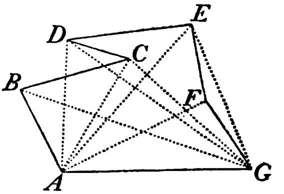


fig. 109.



in which the side AG, together with the angles BAG, CAG, DAG, EAG, FAG, BGA, CGA, DGA, EGA, FGA, are given; from them find its area. The polygon, *fig.* 108, has all its angles outwards, the figure 109 has some of its angles inwards.

1. From the given angles we may find the angles BAC, CAD, DAE, EAF, FAG, likewise ABG, ACG, ADG, AEG, AFG, merely by subtraction. If these are found; then from the triangles BAG, CAG, DAG, EAG, FAG, we obtain the following expressions :

$$AB = \frac{AG \text{ Sin. } BGA}{\text{Sin. } ABG}, \quad AC = \frac{AG \text{ Sin. } CGA}{\text{Sin. } ACG},$$

$$AD = \frac{AG \text{ Sin. } DGA}{\text{Sin. } ADG}, \quad AE = \frac{AG \text{ Sin. } EGA}{\text{Sin. } AEG},$$

$$AF = \frac{AG \text{ Sin. } FGA}{\text{Sin. } AFG}.$$

2. Hence, and from § XXVI, we obtain

$$\triangle BAC = \tfrac{1}{2} AB \,.\, AC \text{ Sin. } BAC = \frac{\tfrac{1}{2} AG^2 \text{ Sin. } BGA \text{ Sin. } CGA \text{ Sin. } BAC}{\text{Sin. } ABG \text{ Sin. } ACG}$$

$$\triangle CAD = \tfrac{1}{2} AC \,.\, AD \text{ Sin. } CAD = \frac{\tfrac{1}{2} AG^2 \text{ Sin. } CGA \text{ Sin. } DGA \text{ Sin. } CAD}{\text{Sin. } ACG \text{ Sin. } ADG}$$

$$\triangle DAE = \tfrac{1}{2} AD \,.\, AE \text{ Sin. } DAE = \frac{\tfrac{1}{2} AG^2 \text{ Sin. } DGA \text{ Sin. } EGA \text{ Sin. } DAE}{\text{Sin. } ADG \text{ Sin. } AEG}$$

$$\triangle EAF = \tfrac{1}{2} AE \,.\, AF \text{ Sin. } EAF = \frac{\tfrac{1}{2} AG^2 \text{ Sin. } EGA \text{ Sin. } FGA \text{ Sin. } EAF}{\text{Sin. } AEG \text{ Sin. } AFG}$$

$$\triangle FAG = \tfrac{1}{2} AF \,.\, AG \text{ Sin. } FAG = \frac{\tfrac{1}{2} AG^2 \text{ Sin. } FGA \text{ Sin. } FAG}{\text{Sin. } AFG}$$

3. Now since the heptagon $ABCDEFG = \triangle BAC \pm \triangle CAD + \triangle DAE + \triangle EAF + \triangle FAG$, (the upper of the two signs $\pm$ obtains for *fig.* 108); then it is =

$$\tfrac{1}{2} AG^2 \left\{ \begin{array}{l} \dfrac{\text{Sin. } BGA \text{ Sin. } CGA \text{ Sin. } BAC}{\text{Sin. } ABG \text{ Sin. } ACG} \\ \pm \dfrac{\text{Sin. } CGA \text{ Sin. } DGA \text{ Sin. } CAD}{\text{Sin. } ACG \text{ Sin. } ADG} \\ + \dfrac{\text{Sin. } DGA \text{ Sin. } EGA \text{ Sin. } DAE}{\text{Sin. } ADG \text{ Sin. } AEG} \\ + \dfrac{\text{Sin. } EGA \text{ Sin. } FGA \text{ Sin. } EAF}{\text{Sin. } AEG \text{ Sin. } AFG} \\ + \dfrac{\text{Sin. } FGA \text{ Sin. } FAG}{\text{Sin. } AFG} \end{array} \right\}$$

4. If we take G, instead of A, for the common vertex of the triangles, we then obtain besides, because $ABCDEFG = \pm \triangle FGE + \triangle EGD \pm \triangle DGC + \triangle CGB + \triangle BGA$, the following expression for the area:

$$\frac{1}{2}AG^2\left\{\begin{array}{l}\pm\dfrac{\text{Sin. } FAG \text{ Sin. } EAG \text{ Sin. } FGE}{\text{Sin. } AFG \text{ Sin. } AEG}\\ +\dfrac{\text{Sin. } EAG \text{ Sin. } DAG \text{ Sin. } EGD}{\text{Sin. } AEG \text{ Sin. } ADG}\\ \pm\dfrac{\text{Sin. } DAG \text{ Sin. } CAG \text{ Sin. } DGC}{\text{Sin. } ADG \text{ Sin. } ACG}\\ +\dfrac{\text{Sin. } CAG \text{ Sin. } BAG \text{ Sin. } CGB}{\text{Sin. } ACG \text{ Sin. } ABG}\\ \qquad +\dfrac{\text{Sin. } BGA \text{ Sin. } BAG}{\text{Sin. } ABG}\end{array}\right\}.$$

Cor. Although the calculation has been effected here for a heptagon, we can readily perceive from hence, how to proceed in the case of every other polygon. The law, which the terms of the expressions in 3 and 4 observe, may be more easily understood by inspection than by words. Moreover, these expressions mutually serve to prove the calculation.

Exam. Let (*fig.* 108) $BAG = 120^\circ$, $CAG = 101^\circ.\ 37'$, $DAG = 81^\circ.\ 20'$, $EAG = 48^\circ.\ 49'$, $FAG = 23^\circ.\ 48'$; $FGA = 88^\circ.\ 14'$, $EGA = 70^\circ$, $DGA = 51^\circ.\ 27'$, $CGA = 34^\circ.\ 7'$, $BGA = 22^\circ.\ 10'$. Here $ABG = 37^\circ.\ 50'$, $ACG = 44^\circ.\ 16'$, $ADG = 47^\circ.\ 13'$, $AEG = 61^\circ.\ 41'$, $AFG = 67^\circ.\ 58'$; $BAC = 18^\circ.\ 23'$, $CAD = 20^\circ.\ 17'$, $DAE = 33^\circ.\ 1'$, $EAF = 24^\circ.\ 31'$; $FGE = 18^\circ.\ 14'$, $EGD = 18^\circ.\ 33'$, $DGC = 17^\circ.\ 20'$, $CGB = 11^\circ.\ 57'$. We consequently have

$$\frac{\text{Sin. } BGA \text{ Sin. } CGA \text{ Sin. } BAC}{\text{Sin. } ABG \text{ Sin. } ACG} = 0{\cdot}1558865$$

$$\frac{\text{Sin. } CGA \text{ Sin. } DGA \text{ Sin. } CAD}{\text{Sin. } ACG \text{ Sin. } ADG} = 0{\cdot}2968328$$

$$\frac{\text{Sin. } DGA \text{ Sin. } EGA \text{ Sin. } DAE}{\text{Sin. } ADG \text{ Sin. } AEG} = 0{\cdot}6197669$$

$$\frac{\text{Sin. } EGA \text{ Sin. } FGA \text{ Sin. } EAF}{\text{Sin. } AEG \text{ Sin. } AFG} = 0{\cdot}4776058$$

$$\frac{\text{Sin. } FGA \text{ Sin. } FAG}{\text{Sin. } AFG} = 0{\cdot}4351331\ ;$$

and $\therefore$, by 3, heptagon $ABCDEFG = 0{\cdot}9926125 \,.\, AG^2$.

Further, we have

$$\frac{Sin.\ FAG\ Sin.\ EAG\ Sin.\ FGE}{Sin.\ AFG\ Sin.\ AEG} = 0{\cdot}1155552$$

$$\frac{Sin.\ EAG\ Sin.\ DAG\ Sin.\ EGD}{Sin.\ AEG\ Sin.\ ADG} = 0{\cdot}3635294$$

$$\frac{Sin.\ DAG\ Sin.\ CAG\ Sin.\ DGC}{Sin.\ ADG\ Sin.\ ACG} = 0{\cdot}5631592$$

$$\frac{Sin.\ CAG\ Sin.\ BAG\ Sin.\ CGB}{Sin.\ ACG\ Sin.\ ABG} = 0{\cdot}4102605$$

$$\frac{Sin.\ BGA\ Sin.\ BAG}{Sin.\ ABG} = 0{\cdot}5327207\ ;$$

and $\therefore$ by 4, heptagon $ABCDEFG = 0{\cdot}9926125 \,.\, AG^2$, as before.

Now, in order to determine fully the area of the heptagon, it is only requisite to know the length of AG. Thus, let $AG = 450$: then heptagon $ABCDEFG = 201004{\cdot}03$ nearly.

COR. This method of calculating the area of a figure, may be very advantageously made use of in its division. Thus, if we wish to cut off $\frac{5}{8}$ths from the heptagon just calculated, by a line drawn from the point A, we have

$$\tfrac{5}{8}\ ABCDEFG = 0{\cdot}6203828 \,.\, AG^2.$$

Now, by the above calculations $\triangle\ BAC = 0{\cdot}1558865 \,.\, AG^2$, $\triangle\ CAD = 0{\cdot}2968328 \,.\, AG^2$, $\triangle\ DAE = 0{\cdot}6197669 \,.\, AG^2$; whence it appears, that the line of section Aa falls between AD and AE, and that $\triangle\ DAa = 0{\cdot}1676635 \,.\, AG^2$. Consequently we have $\triangle\ DAE : \triangle\ DAa = 6197669 : 1676635 = DE : Da$, or approximately $DE : DA = 920 : 257$, which is found by means of Continued Fractions. According to this proportion $\therefore$ the line DE must be divided in a, and then, when Aa is drawn, we have $ABCDa = \frac{5}{8} ABCDEFG$

SECTION C.

PROB. *From the same parts as are given in the foregoing problem, to find any side whatever of the polygon.*

SOLUT. Let *ABCDEFG* (*fig.* 108) be the polygon, in which the same parts are given, as in the foregoing problem: find the sides *AB*, *BC*, *CD*, *DE*, *EF*, *FG*.

In the triangles *ABG*, *AFG*, all the angles and the side *AG* are given; consequently also the sides *AB*, *FG* may be found. The remaining sides can be found by the seventh solution in § XLVIII. Thus, if it is wished to determine the line *DE*, then the comparison with the above solution gives $a = AG$, $\alpha = DAG$, $\beta = EAG$, $\gamma = EGA$, $\delta = DGA$, and we $\therefore$ have

$$DE = AG \sqrt{\left[\left(\frac{Sin.\ DGA}{Sin.\ A}\right)^2 + \left(\frac{Sin.\ EGA}{Sin.\ B}\right)^2 - \frac{2\ Sin.\ DGA\ Sin.\ EGA\ Cos.\ (DAG - EAG)}{Sin.\ A\ Sin.\ B}\right]}$$

in which $A = 180^0 - DAG - EAG$, $B = 180^0 - EGA - EAG$.

REMARK. The following authors treat of Polygonometry: Lexell (De Resolutione Polygonorum Rectilineorum; Commentarii Novi. Petrop. T. 19, 20); L'Huilier (Polygonometrie, ou De la Mesure des Figures Rectilignes, &c. Geneve et Paris, 1789); Neumann (New Contributions to Practical Geometry. Munich, 1800); Däzel in the above-mentioned work; Mascheroni (Problemi di Geometrica. Milano, Anno X (1802), p. 105, et seq.); and also in a particular work (Metodo di misurari i Poligoni Piani), which appeared at Pavia in 1787, and which contains all the problems to be found in L'Huilier's Polygonometry, which appeared in 1789; I have not been able, however, to get a sight of it. On Tetragonometry, in particular, Lambert has treated in his Contributions, part II. p. 175, et seq.; J. T. Mayer (Tetragonometriæ Specimen I. Göt. 1773); Biörnsen (Introductio in Tetragonometriam ad mentem v. c. Lambert Analytice Conscripta Havniæ, 1780). There is also much matter respecting this subject amongst others to be found in Carnot's Geometrie de Position, Paris, 1803, p. 304, &c. which part was printed before the author became acquainted with the above-mentioned work of L'Huilier.

IX. A FEW IMPORTANT PROBLEMS IN PRACTICAL GEOMETRY, CHIEFLY RELATING TO THE MODE OF TREATING NEGATIVE ANGLES.

SECTION CI.

PRELIMINARIES.

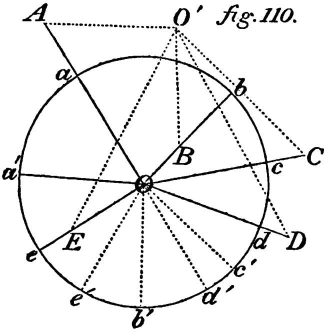


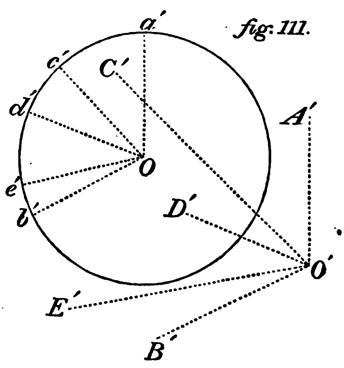
fig. 110.

1. Let A, B, C, D, E, (*fig.* 110) be a system of points; O a point, for which, in the relative position of these points in the assumed figure, and for any calculation founded upon it, we assume, $AOB = \alpha$, $BOC = \beta$, $COD = \gamma$, $DOE = \delta$. Suppose, that after having completed the calculation, we wish to apply any algebraical expression taken from it to a particular case; but that the point O has not the position supposed in the calculation with respect to the system, but is situated in O': then it is evident, that with regard to the absolute magnitude of the angles, we must put, $\alpha = AO'B$, $\beta = BO'C$, $\gamma = CO'D$, $\delta = DO'E$. Now, in order to determine, likewise, which angle, according to the change of position, is to be considered as positive, and which as negative, may be discovered by the method in § LXXXIV, with this difference, that here the corresponding radii coincide with the positions of the lines containing the angles, instead of their being as in the above section, drawn according to the direction in which they are produced. This change takes place merely for this reason, because in the present case we have to do with the angles themselves, while in § LXXXIV their adjacent angles were considered.

2. Let ∴ a circle be described about O, which cuts the lines OA, OB, OC, OD, OE, or these produced in the points a, b, c, d, e: then Oa, Ob, Oc, Od, Oe, are the corresponding radii of these lines. Let now the point O move into O', and let the lines Oa', Ob', Oc', Od', Oe', be drawn parallel to the lines $O'A$, $O'B$, $O'C$, $O'D$, $O'E$, respectively: then the first are the corresponding radii of the latter, with reference to O', and we have $a'Ob' = AO'B$, $b'Oc' = BO'C$, $c'Od' = CO'D$, $d'Oe' = DO'E$. Again, as in § LXXXIV, suppose a moving radius, which turns round the point O; this, in order to move from Oa', into Ob', must have an opposite motion from that which it has when it moves from Oa into Ob. The two angles aOb, $a'Ob'$, and ∴ also AOB, $AO'B$ are consequently opposite to one another in position, and we ∴ have $a = -AO'B$. The same obtains of the angles bOc, $b'Oc'$, and ∴ also of BOC, $BO'C$; consequently $\beta = -BO'C$. On the other hand, the moving radius has the same direction, whether it move from Oc into Od, or from Oc' into Od'; consequently γ is positive, and we have $\gamma = CO'D$. For a similar reason δ is positive, and we ∴ have $\delta = DO'E$.

3. Hitherto it has been assumed, that the point O alone moves, while, on the other hand, the points A, B, C, D, E, retain their places. But even for the case where the points of the system itself change their positions with respect to each other, the above rule admits of no exception. Suppose the points A, B, C, D, E, O, have the position A', B', C', D', E', O', (*fig.* 111), so that A moves into A', B into B', and so on, and that we wish to find the values of α, β, γ, δ, accordingly: then draw the radii Oa', Ob', Oc', Od', Oe', corresponding to the lines $O'A'$, $O'B'$, $O'C'$, $O'D'$, $O'E'$, and compare *fig.* 110 with *fig.* 111. Then we get $\alpha = -a'Ob' = -A'O'B'$, $\beta = b'Oc' = B'O'C'$, $\gamma = -c'Od' = -C'O'D'$, $\delta = -d'Oe' = -D'O'E'$.

fig. 111.



It will not be necessary, in order to distinguish between

the positive and negative position of the angles, actually to draw the corresponding radii. After a little practice, the turning the compasses only will be sufficient, as every person may easily perceive without any diffuse explanation.

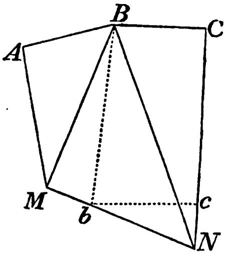
SECTION CII.

PROB. *Two inaccessible points are given in position and distance: determine the positions and distances of two other points with reference to the former, without measuring any line.*

First Solution.

Let A, B, C, (*fig.* 112) be the three points; the distances AB, BC, and the angle ABC are given: find the positions of two other points M, N merely by measuring angles.

fig. 112.



I shall assume, that all the three points A, B, C, can be seen both from M and N, and that at these last points (if we suppose MC, NA drawn), the angles AMB, BMC, ANB, BNC, can be measured. For this case, the points M, N, may be determined singly by § LIV, and the distances MA, MB, MC, NA, NB, NC, calculated. But the distance MN may also be easily computed: for in the triangle MBN, we then have the angle $MBN = ABC - ABM - CBN$, and the sides MB, NB.

Second Solution.

1. I shall now assume, that from M the points A, B, N can be seen, but not the point C, and that from N the points B, C, M, but not the point A. Then the angles AMB, BMN, BNC, CNM, can be measured. Let $AB = a$, $BC = b$, $ABC = B$, $AMB = \alpha$, $BMN = \beta$, $BNC = \gamma$, $CNM = \delta$. If, besides, the angle MAB is known, then all the rest are known. Put $\therefore MAB = \phi$, and draw Bb, bc parallel to the lines CN, BC.

2. From these assumptions we obtain $BbM = CNM = \delta$, $ABM = 180^0 - (\alpha + \phi)$, $MBb = 180^0 - (\beta + \delta)$, and hence $bBC = ABC - ABM - MBb = B + \alpha + \beta + \delta + \phi - 360^0 = Ccb$; consequently $Nbc = Ccb - CNM = B + \alpha + \beta + \phi - 360^0$, $BCN = 180^0 - bBC = 540^0 - (B + \alpha + \beta + \delta + \phi)$, $CBN = 180^0 - BCN - BNC = B + \alpha + \beta + \gamma + \delta + \phi - 360^0$. If $\therefore$ for shortness' sake, we put $B + \alpha + \beta - 360^0 = \kappa$, $B + \alpha + \beta - \gamma + \delta - 360^0 = \lambda$: then $Nbc = \kappa + \phi$, $CBN = \gamma + \phi$.

3. The triangle ABM gives

$$BM = \frac{AB \; Sin.\; MAB}{Sin.\; AMB} = \frac{a \; Sin.\; \phi}{Sin.\; \alpha},$$

and the triangle MBb,

$$Bb = \frac{BM \; Sin.\; BMN}{Sin.\; BbM} = \frac{a \; Sin.\; \phi \; Sin.\; \beta}{Sin.\; \alpha \; Sin.\; \delta};$$

further, the triangle Nbc,

$$Nc = \frac{bc \; Sin.\; Nbc}{Sin.\; CNM} = \frac{b \; Sin.\; (\kappa + \phi)}{Sin.\; \delta},$$

and the triangle BNC,

$$CN = \frac{BC \; Sin.\; CBN}{Sin.\; BNC} = \frac{b \; Sin.\; (\lambda + \phi)}{Sin.\; \gamma}.$$

4. Now $CN = Cc + Nc = Bb + Nc$: we $\therefore$ have the equation

$$\frac{b \; Sin.\; (\lambda + \phi)}{Sin.\; \gamma} = \frac{a \; Sin.\; \beta \; Sin.\; \phi}{Sin.\; \alpha \; Sin.\; \delta} + \frac{b \; Sin.\; (\kappa + \phi)}{Sin.\; \delta}$$

or, when we expand $Sin.\; (\lambda + \phi)$, $Sin.\; (\kappa + \phi)$, divide by $Sin.\; \phi$, and substitute $Cot.\; \phi$ for $\frac{Cos.\; \phi}{Sin.\; \phi}$, we get

$$\frac{b \; (Sin.\; \lambda \; Cot.\; \phi + Cos.\; \lambda)}{Sin.\; \gamma} = \frac{a \; Sin.\; \beta}{Sin.\; \alpha \; Sin.\; \delta} + \frac{b \; (Sin.\; \kappa \; Cot.\; \phi + Cos.\; \kappa)}{Sin.\; \delta},$$

whence we obtain

$$Cot.\ \phi = \frac{a\ Sin.\ \beta\ Sin.\ \gamma + b\ Sin.\ \alpha\ (Sin.\ \gamma\ Cos.\ \kappa - Sin.\ \delta\ Cos.\ \lambda)}{b\ Sin.\ \alpha\ (Sin.\ \delta\ Sin.\ \lambda - Sin.\ \gamma\ Sin.\ \kappa)}.$$

EXAM. 1. Let $a = 351$, $b = 402$, $B = 167^0.\ 4'$, $\alpha = 33^0.\ 3'$, $\beta = 90^0.\ 51'$, $\gamma = 27^0.\ 19'$, $\delta = 76^0.\ 20'$; $\therefore$ $\kappa = -69^0.\ 2'$, $\lambda = -20^0.\ 1'$, and consequently $Sin.\ \kappa = -Sin.\ 69^0.\ 2'$, $Cos.\ \kappa = Cos\ 69^0.\ 2'$, $Sin.\ \lambda = -Sin.\ 20^0.\ 1'$, $Cos.\ \lambda = Cos.\ 20^0.\ 1'$. We $\therefore$ have

$$\begin{aligned}
a\ Sin.\ \beta\ Sin.\ \gamma &= 161{\cdot}0590\\
b\ Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \kappa &= 36{\cdot}0009\\
b\ Sin.\ \alpha\ Sin.\ \delta\ Cos.\ \lambda &= 200{\cdot}1631\\
b\ Sin.\ \alpha\ Sin.\ \delta\ Sin\ \lambda &= -\ 71{\cdot}9193\\
b\ Sin.\ \alpha\ Sin\ \gamma\ Sin.\ \kappa &= -\ 93{\cdot}9490
\end{aligned}$$

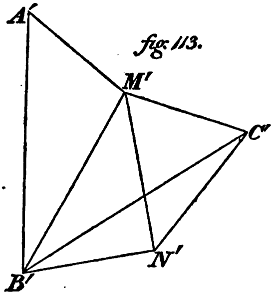
and $\therefore$,

$$Cot.\ \phi = \frac{161{\cdot}0590 + 36{\cdot}0009 - 200{\cdot}1631}{-\ 72{\cdot}9193 + 93{\cdot}9490}$$

$$= -\ 0{\cdot}1475627;$$

consequently $\phi = 98^0.\ 23'.\ 39''$.

EXAM. 2. I shall assume, that the points A, B, C, M, N, have the positions A', B', C', M', N', (*fig.* 113). Here we must first of all ascertain which of the given angles, agreeably to this change in the positions, we are to consider as positive, and which as negative. In the first place, suppose two circles described about M, M'; then we shall immediately perceive, that the corresponding radii of MA, MB have an opposite position from that which the corresponding radii of $M'A'$, $M'B'$ have, and that this is likewise the case with the corresponding radii of MB, MN, compared with those of $M'B'$, $M'N'$; consequently both α and β are negative. In the



same manner it may be shown, if two circles are described about B, B', that B is likewise negative. If, on the other hand, two circles are described about N, N', then it is evident, that the angles BNC, $B'N'C'$, also CNM, $C'N'M'$, have the same position, and ∴ that γ and δ remain positive.

Now, let $B = -A'B'C' = -56^0.\ 1'$, $\alpha = -A'M'B' = -103^0.\ 6'$, $\beta = -B'M'N' = -41^0.\ 8'$, $\gamma = B'N'C' = 144^0.\ 27'$, $\delta = C'N'M' = 50^0.\ 43'$, $a = A'B' = 980$, $b = B'C' = 1000$. Here $\kappa = B + \alpha + \beta - 360^0 = -560^0.\ 15'$, $\lambda = B + \alpha + \beta - \gamma + \delta - 360^0 = -653^0.\ 59'$; consequently $Sin.\ \kappa = Sin. -560^0.\ 15' = -Sin.\ 560^0.\ 15' = -Sin.\ 200^0.\ 15' - Sin.\ 20^0.\ 15'$, $Cos.\ \kappa = Cos. -560^0.\ 15' = Cos.\ 560^0.\ 15' = Cos.\ 200^0.\ 15' = -Cos.\ 20^0.\ 15'$, $Sin.\ \lambda = Sin. -653^0.\ 59' = -Sin.\ 653^0.\ 59' = -Sin.\ 293^0.\ 59' = Sin.\ 66^0.\ 1'$, $Cos.\ \lambda = Cos. -653^0.\ 59' = Cos.\ 653^0.\ 59' = Cos.\ 293^0.\ 59' = Cos.\ 66^0.\ 1'$; hence

$$\begin{aligned} a\ Sin.\ \beta\ Sin.\ \gamma &= -\ 374{\cdot}8122 \\ b\ Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \kappa &= \ 531{\cdot}2812 \\ b\ Sin.\ \alpha\ Sin.\ \delta\ Cos.\ \lambda &= -\ 306{\cdot}4307 \\ b\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \lambda &= -\ 688{\cdot}7940 \\ b\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \kappa &= -\ 196{\cdot}0000. \end{aligned}$$

We have ∴

$$Cot.\ \phi = \frac{-\ 374{\cdot}8122 + 531{\cdot}2812 + 306{\cdot}4307}{-\ 688{\cdot}7940 + 196{\cdot}0000}$$

$$= \frac{+\ 462{\cdot}8997}{-\ 492{\cdot}7940} = -\ 0{\cdot}9393371,$$

and consequently either $\phi = 133^0.\ 12'.\ 30''$, or $\phi = -46^0.\ 47'.\ 30''$. The first of these two values for ϕ cannot be made use of here, because $B'A'M' + A'M'B'$ cannot be greater than 180^0. We ∴ merely have

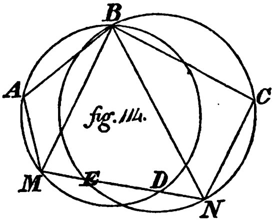
$$\phi = -\ 46^0.\ 47'.\ 30''.$$

Since ϕ is here negative, consequently $A'M'$, as the figure shows, must fall beyond the line $A'B'$.

When $B'A'M'$ is found, then all the remaining parts of the figure may likewise be calculated.

Cor. The two required points may also be found by the following very simple notation.

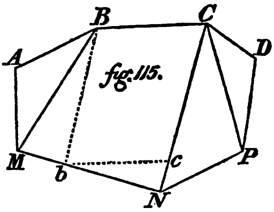
Let A, B, C (*fig.* 114), be the three given points. Upon AB as a chord, describe a circle $BDMA$, which includes the angle α, and upon BC another circle $CNEB$, which includes the angle γ; then take the arc $BD = 2\beta$, and the arc $CBE = 2\delta$, and through the points thus determined, viz., D, E, draw the line DE, which produced meets the two circles in M, N; then M, N are the two points sought. For since $BD = 2\beta$, and $CBE = 2\delta$: then $BMD = \beta$, $CNE = \delta$; also immediately from the construction itself, $AMB = \alpha$, $BNC = \gamma$.



SECTION CIII.

Prob. *Four inaccessible points are given in position and distance: determine the position of three other points in reference to these, without measuring any line.*

Solut. Let A, B, C, D (*fig.* 115) be the four given points, and M, N, P the three whose position it is required to be determined. If now from each of the points sought, three of the given ones are seen; then these may be determined singly by § 54. Since this case involves no difficulty, I shall ∴ assume, that from M the three points A, B, N only are seen, and from P the three points D, C, N only; but from N only one of the given points, say C, and the two required ones M, P. On this supposition we can ∴ measure the angles AMB, BMN, CNM, CNP, CPN, CPD. Further, since the four points A, B, C, D, are given in position and distance, we therefore have likewise the lines AB, BC, CD, and the angles ABC, BCD. Let ∴ $AB = a$, $BC = b$, $CD = c$, $ABC = B$, $BCD = C$, $AMB = \alpha$, $BMN = \beta$, $CNM = \gamma$, $CNP = \delta$,



$CPN = \varepsilon$, $CPD = \zeta$. If besides we had found the angle MAB; then all the rest of the figures would be known. Put $\therefore$ $MAB = \phi$, and draw Bb parallel to CN, and bc parallel to BC.

1. Then $ABM = 180^0 - (\alpha + \phi)$, $BbM = CNM = \gamma$, and $\therefore$ $MBb = 180^0 - BMb - BbM = 180^0 - (\beta + \gamma)$; consequently $Ccb = bBC = ABC - ABM - MBb = B + \alpha + \phi + \beta + \gamma - 360^0$, $Nbc = Ccb - CNM = B + \alpha + \phi + \gamma - 360^0$. Further, since in the heptagon $ABCDPNM$ the sum of all the angles $= 10\,R = 900^0$; therefore $CDP = 900^0 - (AMN + MNP + NPD + BCD + ABC + MAB) = 900^0 - (\alpha + \beta + \gamma + \delta + \varepsilon + \zeta + C + B + \phi)$. If we abbreviate by putting $B + \alpha + \beta - 360^0 = \kappa$, $900^0 - (\alpha + \beta + \gamma + \delta + \varepsilon + \zeta + B + C) = \lambda$; then $Nbc = \kappa + \phi$, $CDP = \lambda - \phi$.

2. In the triangle ABM,

$$BM = \frac{AB \text{ Sin. } MAB}{\text{Sin. } AMB} = \frac{a \text{ Sin. } \phi}{\text{Sin. } \alpha};$$

in the triangle MBb,

$$Bb = \frac{BM \text{ Sin. } BMb}{\text{Sin. } BbM} = \frac{a \text{ Sin. } \phi \text{ Sin. } \beta}{\text{Sin. } \alpha \text{ Sin. } \gamma};$$

in the triangle Nbc,

$$Nc = \frac{bc \text{ Sin. } Nbc}{\text{Sin. } bNc} = \frac{b \text{ Sin. } (\kappa + \phi)}{\text{Sin. } \gamma};$$

in the triangle CDP,

$$CP = \frac{CD \text{ Sin. } CDP}{\text{Sin. } CPD} = \frac{c \text{ Sin. } (\lambda - \phi)}{\text{Sin. } \zeta},$$

and in the triangle CPN,

$$CN = \frac{CP \text{ Sin. } CPN}{\text{Sin. } CNP} = \frac{c \text{ Sin. } (\lambda - \phi) \text{ Sin. } \varepsilon}{\text{Sin. } \zeta \text{ Sin. } \delta}.$$

3. Now since $CN = Cc + Nc = Bb + NC$: therefore we have

the equation

$$\frac{c\ Sin.\ (\lambda-\phi)\ Sin.\ \varepsilon}{Sin.\ \zeta\ Sin.\ \delta}=\frac{a\ Sin.\ \phi\ Sin.\ \beta}{Sin.\ \alpha\ Sin.\ \gamma}+\frac{b\ Sin.\ (\kappa+\phi)}{Sin.\ \gamma}.$$

Expand $Sin.\ (\lambda-\phi)$, $Sin.\ (\kappa+\phi)$, divide the equation by $Sin.\ \phi$, and put $Cot.\ \phi$ for $\frac{Cos.\ \phi}{Sin.\ \phi}$; this gives

$$\frac{c\ Sin.\ \varepsilon\ (Sin.\ \lambda\ Cot.\ \phi-Cos.\ \lambda)}{Sin.\ \zeta\ Sin.\ \delta}=\frac{a\ Sin.\ \beta}{Sin.\ \alpha\ Sin.\ \gamma}$$
$$+\frac{b\ (Sin.\ \kappa\ Cot.\ \phi+Cos.\ \kappa)}{Sin.\ \gamma},$$

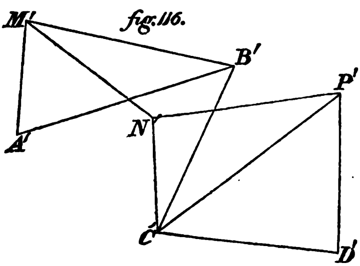
and hence we obtain,

$$Cot.\ \phi=$$
$$\frac{\left[\begin{array}{c}a\ Sin.\ \beta\ Sin.\ \delta\ Sin.\ \zeta+b\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \zeta\ Cos.\ \kappa\\ +c\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \varepsilon\ Cos.\ \lambda\end{array}\right]}{-b\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \zeta\ Sin.\ \kappa+c\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \varepsilon\ Sin.\ \lambda},$$

whence the angle ϕ, and consequently all the others, may be found.

EXAM. A', B', C', D', (*fig.* 116) are four points, whose position is known; M', N', P' three others, whose position is sought. From M' only A', B', N', are visible, from N' only C', M', P', and from P' only C', D', N'. Let the angles measured from these points be as follow: $A'M'B'=80^{0}.\ 8'$, $B'M'N'=24^{0}.\ 55'$, $C'N'M=124^{0}.\ 16'$, $C'N'P'=98^{0}.\ 44'$, $C'P'N'=29^{0}.\ 13'$, $C'P'D'=51^{0}.\ 19'$. For the points A', B', C', D', we have $A'B'=815$, $B'C'=670$, $C'D'=660$, $A'B'C'=49^{0}.\ 54'$, $B'C'D'=73^{0}.\ 57'$.

fig. 116.



The comparison of *figs.* 115 and 116 shows immediately their relation, and then, when what has been said in § 101, is applied to every two corresponding angular points, we have

$\alpha = -80^\circ.\ 8'$, $\beta = 24^\circ.\ 55'$, $\gamma = -124^\circ.\ 16'$, $\delta = -98^\circ.\ 44'$, $\varepsilon = -29^\circ.\ 13'$, $\zeta = -51^\circ.\ 19'$, $B = 49^\circ.\ 54'$, $C = -73^\circ.\ 57'$; likewise $a = 815$, $b = 670$, $c = 660$. From these data we obtain $\kappa = -365^\circ.\ 19'$, $\lambda = 1282^\circ.\ 48'$; $\therefore$ *Sin.* κ = *Sin.* $-365^\circ.\ 19' = -$*Sin.* $365^\circ.\ 19' = -$*Sin.* $5^\circ.\ 19'$, *Cos.* κ = *Cos.* $-365^\circ.\ 19'$ = *Cos.* $365^\circ.\ 19'$ = *Cos.* $5^\circ.\ 19'$, *Sin.* λ = *Sin.* $1282^\circ.\ 48'$ = *Sin.* $202^\circ.\ 48' = -$ *Sin.* $22^\circ.\ 48'$. *Cos.* λ = *Cos.* $1282^\circ.\ 48'$ = *Cos.* $202^\circ.\ 48' = -$*Cos.* $22^\circ.\ 48'$. Further, *Sin.* $\alpha = -$ *Sin.* $80^\circ.\ 8'$, *Sin.* β = *Sin.* $24^\circ.\ 55'$, *Sin.* $\gamma = -$ *Sin.* $124^\circ.\ 16' = -$ *Sin.* $55^\circ.\ 44'$, *Sin.* $\delta = -$ *Sin.* $98^\circ.\ 44' = -$*Sin.* $81^\circ.\ 16'$, *Sin.* $\varepsilon = -$ *Sin.* $29^\circ.\ 13'$, *Sin.* $\zeta = -$*Sin.* $51^\circ.\ 19'$. We consequently have

$$\begin{aligned}
a\ Sin.\ \beta\ Sin.\ \delta\ Sin.\ \zeta &= 264{\cdot}9228\\
b\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \zeta\ Cos.\ \kappa &= -507{\cdot}1091\\
c\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \varepsilon\ Cos.\ \lambda &= 241{\cdot}8042\\
b\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \zeta\ Sin.\ \kappa &= 47{\cdot}1919\\
c\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \varepsilon\ Sin.\ \lambda &= 101{\cdot}6451\ ;
\end{aligned}$$

and $\therefore$

$$Cot.\ \phi = \frac{264{\cdot}9228 - 507{\cdot}1091 + 241{\cdot}8042}{-47{\cdot}1919 + 101{\cdot}6451} = \frac{-0{\cdot}3821}{54{\cdot}4532}$$
$$= -0{\cdot}0070170.$$

We consequently have, either, $\phi = 90^\circ.\ 24'.\ 7''$, or $= -89^\circ.\ 35'.\ 53''$. In the figure, for which this calculation has been made, ϕ has the second value, and $\therefore$ $B'A'M' = 89^\circ.\ 35'.\ 53''$.

In order to prove the calculation, it is merely requisite to substitute the value here obtained for ϕ in the equation

$$\frac{c\ Sin.\ (\lambda - \phi)\ Sin.\ \varepsilon}{Sin.\ \zeta\ Sin.\ \delta} = \frac{a\ Sin.\ \phi\ Sin.\ \beta}{Sin.\ \alpha\ Sin.\ \gamma} + \frac{b\ Sin.\ (\kappa + \phi)}{Sin.\ \gamma}$$

which was found in Solution 3. Since this proof contains some important information, I shall give it here.

Since $\lambda = 1282^\circ.\ 48'$, $\kappa = -365^\circ.\ 19'$, $\phi = -89^\circ.\ 35'.\ 53''$: therefore $\lambda - \phi = 1372^\circ.\ 23'.\ 53''$, $\kappa + \phi = -454^\circ.\ 54'.\ 53''$; $\therefore$ *Sin.* $(\lambda - \phi)$ = *Sin.* $1372^\circ.\ 23'.\ 53''$ = *Sin.* $292^\circ.\ 37'.\ 53''$ $= -$*Sin.* $67^\circ.\ 36'.\ 7''$, *Sin.* $(\kappa + \phi)$ = *Sin.* $-454^\circ.\ 54'.\ 53'' =$ $-$ *Sin.* $94^\circ.\ 54'.\ 53'' = -$ *Sin.* $85^\circ.\ 5'.\ 7''$. We $\therefore$ have,

$$\frac{c\ Sin.\ (\lambda - \phi)\ Sin.\ \varepsilon}{Sin.\ \zeta\ Sin.\ \delta} = 386{\cdot}0368$$

$$\frac{a\ Sin.\ \phi\ Sin.\ \beta}{Sin.\ \alpha\ Sin.\ \gamma} = -\ 421{\cdot}7018$$

$$\frac{b\ Sin.\ (\kappa + \phi)}{Sin.\ \gamma} = 807{\cdot}7391.$$

Thus .·. $386{\cdot}0368 = -\ 421{\cdot}7018 + 807{\cdot}7391$; and this is correct within 0·0005, which error arises from the incorrectness of the Tables.

When the angle ϕ has been found, it is easy to calculate all the lines of the figure. The best mode of proceeding will be, first to derive the general expressions for the required lines from *fig.* 115, and these last from *fig.* 116. Thus if we wish to calculate the line $M'N'$ (*fig.* 116), first of all we find the general expression for MN (*fig.* 115). By Solution 1, $Nbc = \kappa + \phi$, .·. $Ncb = 180^0 - bNc - Nbc = 180^0 - (\gamma + \kappa + \phi)$, and $MBb = 180^0 - (\beta + \gamma)$: we consequently have,

$$Mb = \frac{BM\ Sin.\ MBb}{Sin.\ MbB} = \frac{BM\ Sin.\ (\beta + \gamma)}{Sin.\ \gamma}$$

$$= \frac{a\ Sin.\ \phi\ Sin.\ (\beta + \gamma)}{Sin.^2\ \gamma},$$

$$Nb = \frac{bc\ Sin.\ Ncb}{Sin.\ bNc} = \frac{b\ Sin.\ (\gamma + \kappa + \phi)}{Sin.\ \gamma},$$

and .·.

$$MN = Mb + Nb =$$

$$\frac{a\ Sin.\ \phi\ Sin.\ (\beta + \gamma)}{Sin.\ {}^2\gamma} + \frac{b\ Sin.\ (\gamma + \kappa + \phi)}{Sin.\ \gamma}.$$

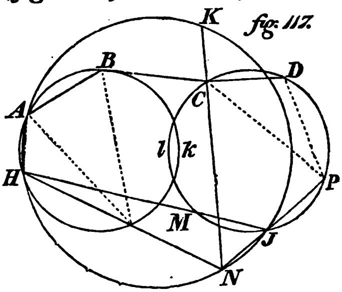
If we now apply the expression to *fig.* 116, we get $a = 815$, $b = 670$, $Sin.\ \phi = -\ Sin.\ 89^0.\ 25'.\ 53''$, $Sin.\ \gamma = -\ Sin.\ 124^0.\ 16' = -\ Sin.\ 55^0.\ 44'$, $Sin.\ (\beta + \gamma) = Sin.\ -\ 99^0.\ 21' = -\ Sin.\ 80^0.\ 39'$, $Sin.\ (\gamma + \kappa + \phi) = Sin. - 579^0.10.'53'' = - Sin.\ 219^0.10.'53'' = Sin.\ 39^0.10.'53''$. Hence we obtain,

$$\frac{a\ Sin.\ \phi\ Sin.\ (\beta + \gamma)}{Sin.\ \gamma^2} = 1177{\cdot}4171$$

$$\frac{b \; Sin. (\gamma + \kappa + \phi)}{Sin. \; \gamma} = -512{\cdot}1946,$$

and $\therefore M'N'$ (*fig.* 116) $= 665{\cdot}2225$

CONST. Upon AB, CD, (*fig.* 117) as chords, describe two circles, so that the arc BkA contain the angle α, and the arc ClD the angle ζ; then assume the arc $BkH = 2\beta$, and the arc $ClJ = 2\varepsilon$, draw HI, and upon this line as a chord describe a circle such, that the arc HNI may contain the angle $\gamma + \delta$: then make the arc $HK = 2\gamma$, or $IK = 2\delta$, and through K, C, draw the line KC, which produced cuts the circle in N. Now draw the lines HN, IN, which cut the other two circles in M, P: then M, N, P, are the three parts sought. Thus we have $AMB = \alpha$, $BMN = \beta$, $CNM = \gamma$, $CNP = \delta$, $CPN = \varepsilon$, $CPD = \zeta$.

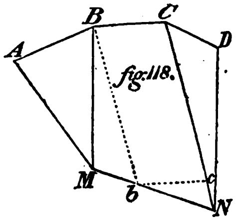


The proof of this construction is founded on the rule, that the angle at the circumference is equal to half the arc upon which it stands, and is easily found.

SECTION CIV.

PROB. *Four points are given in position and distance: required to find the positions and distances of two other points, from each of which, besides the other point sought, only two of those given are seen.*

SOLUT. Let A, B, C, D, (*fig.* 118), be the four given points, M, N, the two sought: from M only A, B, N, and from N only C, D, M, are seen. At M, N, measure the angles $AMB = \alpha$, $BMN = \beta$, $CNM = \gamma$, $CND = \delta$. Further, there are given $AB = a$, $BC = b$, $CD = c$, $ABC = B$, $BCD = C$.



1. If the angle MAB is known; then we know all the rest. Put $\therefore$ $MAB = \phi$, and draw BC parallel to CN, and bc to BC. Now, if we compare *fig.* 118 with *fig.* 115, then we perceive at the first glance, that when we take the triangle CDN from the former, and the quadrilateral $CDPN$ from the latter, all the remainders, even the expressions for the angles and sides, agree. Consequently, as in the foregoing section, when for shortness-sake we put $B + \alpha + \beta - 360^0 = \kappa$, we obtain here also,

$$Bb = \frac{a \, Sin. \, \phi \, Sin. \, \beta}{Sin. \, \alpha \, Sin. \, \gamma}, \; Nc = \frac{b \, Sin. \, (\kappa + \phi)}{Sin. \, \gamma},$$

and $\therefore$

$$NC = \frac{a \, Sin. \, \phi \, Sin. \, \beta}{Sin. \, \alpha \, Sin. \, \gamma} + \frac{b \, Sin. \, (\kappa + \phi)}{Sin. \, \gamma}.$$

2. In the hexagon $ABCDNM$ (*fig.* 118) all the angles taken together $= 8\,R = 720^0$. We $\therefore$ have $CDN = 720^0 - (\alpha + \beta + \gamma + \delta + B + C + \phi)$; or, when we put $720^0 - (\alpha + \beta + \gamma + \delta + B + C) = \lambda$, $CDN = \lambda - \phi$. The triangle CDN consequently gives

$$CN = \frac{c \, Sin. \, (\lambda - \phi)}{Sin. \, \delta}.$$

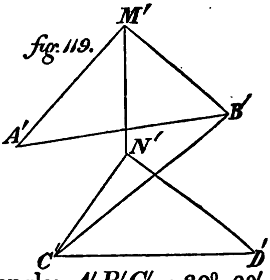
3. If we put the two expressions found for CN in 2, 3, equal to one another, we obtain the equation,

$$\frac{c \, Sin. \, (\lambda - \phi)}{Sin. \, \delta} = \frac{a \, Sin. \, \phi \, Sin. \, \beta}{Sin. \, \alpha \, Sin. \, \gamma} + \frac{b \, Sin. \, (\kappa + \phi)}{Sin. \, \gamma}$$

and this equation, when we treat it as we did a similar one in § CIII, gives

$$Cot. \, \phi = \frac{a \, Sin. \, \beta \, Sin. \, \delta + b \, Sin. \, \alpha \, Sin. \, \delta \, Cos. \, \kappa + c \, Sin. \, \alpha \, Sin. \, \gamma \, Cos. \, \lambda}{-b \, Sin. \, \alpha \, Sin. \, \delta \, Sin. \, \kappa + c \, Sin. \, \alpha \, Sin. \, \gamma \, Sin. \, \lambda}.$$

Exam. Let A', B', C', D', (*fig.* 119), be the four given



points, and M', N', the two points whose position is sought, and of which it is assumed, that from M' only A', B', N', and from N' only C', D', M', are visible. The angles when measured are as follow: $A'M'B' = 86^\circ.\ 54'$, $B'M'N' = 46^\circ.\ 51'$, $C'N'M' = 138^\circ.\ 21'$, $C'N'D' = 87^\circ.\ 25'$. The following lines are given viz. $A'B' = 760$, $B'C' = 800$, $C'D' = 720$, and the angles $A'B'C' = 30^\circ.\ 20'$, $B'C'D' = 39^\circ.\ 34'$.

The comparison of *fig.* 119 with *fig.* 118, immediately shows their relation, and hence we get $a = 760$, $b = 800$, $c = 720$, $B = 30^\circ.\ 20'$, $C = -39^\circ.\ 34'$, $\alpha = -86^\circ.\ 54'$, $\beta = 46^\circ.\ 51'$, $\gamma = -138^\circ.\ 21'$, $\delta = -87^\circ.\ 25'$. For this example $\therefore$ $\kappa = -369^\circ.\ 43'$, $\lambda = 995^\circ.\ 3'$, and consequently $Sin.\ \kappa = Sin.\ -369^\circ.\ 43' = -Sin.\ 369^\circ.\ 43' = -Sin.\ 9^\circ.\ 43'$, $Cos.\ \kappa = Cos.\ -369^\circ.\ 43'. = Cos.\ 369^\circ.\ 43' = Cos.\ 9^\circ.\ 43'$, $Sin.\ \lambda = Sin.\ 995^\circ.\ 3' = Sin.\ 275^\circ.\ 3' = -Sin.\ 84^\circ.\ 57'$, $Cos.\ \lambda = Cos.\ 995^\circ.\ 3' = Cos.\ 275^\circ.\ 3' = Cos.\ 84^\circ.\ 57'$. The calculation, when continued, gives

$$\begin{aligned}
a\ Sin.\ \beta\ Sin.\ \delta &= -553{\cdot}9063\\
b\ Sin.\ \alpha\ Sin.\ \delta\ Cos.\ \kappa &= 786{\cdot}5693\\
c\ Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \lambda &= 42{\cdot}0580\\
b\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \kappa &= -134{\cdot}6863\\
c\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \lambda &= -475{\cdot}9416.
\end{aligned}$$

We $\therefore$ have

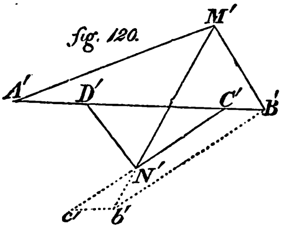
$$Cot.\ \phi = \frac{-553{\cdot}9063 + 786{\cdot}5693 + 42{\cdot}0580}{134{\cdot}6863 - 475{\cdot}9416} = \frac{+274{\cdot}7210}{-341{\cdot}2553}$$
$$= -0{\cdot}8050307.$$

Consequently either $\phi = 128^\circ.\ 50'.\ 6''$, or $\phi = -51^\circ.\ 9'.\ 54''$. The first of these two values cannot obtain here, for otherwise $A'M'B' + M'A'B'$ would be greater than 180°. We consequently have $\phi = -51^\circ.\ 9'.\ 54''$, and $\therefore$ $M'A'B' = 51^\circ.\ 9'.\ 54''$.

When the angle $M'A'B'$ is found, it is easy to determine

the remaining angles of the figure which are not known. Thus $A'B'M' = 180^0 - (A'M'B + B'A'M') = 41^0.\ 56'.\ 6''$, $C'B'M' = A'B'M' + A'B'C' = 72^0.\ 16'.\ 6''$. Now, since in the quadrilateral $B'C'N'M'$ the convex angle $C'N'M'$ $= 221^0.\ 39'$, $B'M'N = 46^0.\ 51'$, $C'B'M' = 72^0.\ 16'.\ 6''$; then $B'C'N' = 19^0.\ 13'.\ 54''$, $\therefore D'C'N' = B'C'N' + B'C'D'$ $= 58^0.\ 47'.\ 54''$, and $C'D'N' = 180^0 - (D'C'N' + C'N'D')$ $= 33^0.\ 47'.\ 6''$.

EXAM. 2. Let A', B', C', D', (*fig.* 120) be four points in a given straight line, whose distance from each other are given, so that $AB = 840$, $BC = 110$, $CD = 500$. M' and N' are two points, whose position is sought; from M' only A', B', N', and from N' only C', D', M, are seen. According to this hypothesis, the angles, which the visible points make with each other, are measured at M', N', and we have found, that $A'M'B' = 105^0.\ 47'$, $B'M'N' = 65^0.\ 31'$, $C'N'M$ $= 35^0.\ 54'$, $C'N'D' = 91^0.\ 35'$.



By comparing *fig.* 120 with *fig.* 118, we get $a = 840$, $b = 110$, $c = 500$, $B = 0$, $C = 180^0$, $\alpha = -105^0.\ 47'$, $\beta = 65^0.\ 31'$, $\gamma = 35^0.\ 34'$, $\delta = -91^0.\ 35'$. We $\therefore$ have $\kappa = -400^0.\ 16'$, $\lambda = 636^0.\ 17'$, and consequently $Sin\ \kappa =$ $- Sin.\ 400^0.\ 16' = - Sin.\ 40^0.\ 16'$, $Cos.\ \kappa = Cos.\ 400^0.\ 16'$, $= Cos.\ 40^0.\ 16'$, $Sin.\ \lambda = Sin.\ 276^0.\ 17' = - Sin.\ 83^0.\ 43'$, $Cos.\ \lambda = Cos.\ 276^0.\ 17' = Cos.\ 83^0.\ 43'$. Further

$$\begin{aligned}
a\ Sin.\ \beta\ Sin.\ \delta &= -764{\cdot}1770\\
b\ Sin.\ \alpha\ Sin.\ \delta\ Cos.\ \kappa &= 80{\cdot}7395\\
c\ Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \lambda &= -30{\cdot}6293\\
b\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \kappa &= -68{\cdot}3913\\
c\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \lambda &= -278{\cdot}1787
\end{aligned}$$

We $\therefore$ have,

$$Cot.\ \phi = \frac{-764{\cdot}1770 + 80{\cdot}7395 - 30{\cdot}6293}{68{\cdot}3913 + 278{\cdot}1787}$$
$$= -2{\cdot}0603826,$$

consequently, either $\phi = 154^0.\ 6'.\ 38''$, or $\phi = -25^0.\ 35'.\ 22''$. The first of these two values cannot be used here, because otherwise $B'A'M' + A'M'B' > 180^0$. Therefore $B'A'M' = 25^0.\ 53'.\ 22''$, whence all the rest of the figure can be determined.

COR. We can always, if we think it advisable, from the formula already calculated for any figure, derive a particular formula for every similar figure. In order to elucidate this by an example, I shall suppose, that we wish, from *fig.* 120, to find a formula for the angle $B'A'M'$, which obtains for all similar figures. Assume, then, $A'M'B' = \alpha'$, $B'M'N' = \beta'$, $C'N'M' = \gamma'$, $C'N'D' = \delta'$, $B'A'M' = \phi'$; comparing it with *fig.* 118 we get $\alpha = -\alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, $\delta = -\delta'$, $B = 0$, $C = 180^0$. We ∴ have $\kappa = -\alpha' + \beta' - 360^0$, $\lambda = 720^0 - (-\alpha' + \beta' + \gamma' - \delta' + 180^0) = 540^0 + \alpha' - \beta' - \gamma' + \delta'$. If these values be substituted in the equation in 3, we then obtain

$$-\,Cot.\,\phi' = \frac{-a\,Sin.\,\beta'\,Sin.\,\delta' + b\,Sin.\,\alpha'\,Sin.\,\delta'\,Cos.\,\kappa - c\,Sin.\,\alpha'\,Sin.\,\gamma'\,Cos.\,\lambda}{-b\,Sin.\,\alpha'\,Sin.\,\delta'\,Sin.\,\kappa - c\,Sin.\,\alpha'\,Sin.\,\gamma'\,Sin.\,\lambda},$$

or

$$Cot.\,\phi' = \frac{-a\,Sin.\,\beta'\,Sin.\,\delta' + b\,Sin.\,\alpha'\,Sin.\,\delta'\,Cos.\,\kappa - c\,Sin.\,\alpha'\,Sin.\,\gamma'\,Cos.\,\lambda}{b\,Sin.\,\alpha'\,Sin.\,\delta'\,Sin.\,\kappa + c\,Sin.\,\alpha'\,Sin.\,\gamma'\,Sin.\,\lambda},$$

We can also find this formula immediately from the figure itself. With this view draw $B'b'$ parallel to $C'N'$, which meets $M'N'$ produced in b', and from b' draw $b'c'$ parallel to $B'C'$, which meets $C'N'$ produced in c'. Then $M'b'B = M'N'C' = \gamma'$; consequently $M'B'b' = 180^0 - (\beta' + \gamma')$. Further, $M'B'A' = 180^0 - (\alpha' + \phi')$, and ∴ $A'B'b' = C'c'b' = D'C'N' = M'B'b' - M'B'A' = \alpha' + \phi' - \beta' - \gamma'$, and $N'b'c' = 180^0 - C'c'b' - b'N'c' = 180^0 - \alpha' + \beta' + \gamma' - \phi' - \gamma' = 180^0 - \alpha' + \beta' - \phi'$. Also $C'D'N' = 180^0 - D'C'N' - C'N'D' = 180^0 - \alpha' - \phi' + \beta' + \gamma' - \delta'$. If ∴ we put $180^0 - \alpha' + \beta' = \kappa'$, $180^0 - \alpha' + \beta' + \gamma' - \delta' = \lambda'$: then $N'b'c = \kappa' - \phi'$, $C'D'N' = \lambda' - \phi'$.

Hence in the triangle $b'N'c'$, we obtain

$$N'c' = \frac{b'c' \; Sin. \; N'b'c'}{Sin. \; b'N'c'} = \frac{b \; Sin. \; (\kappa' - \phi')}{Sin. \; \gamma'},$$

in the triangle $A'M'B'$,

$$B'M' = \frac{A'B' \; Sin. \; B'A'M'}{Sin. \; M'B'A'} = \frac{a \; Sin. \; \phi'}{Sin. \; \alpha'},$$

in the triangle $M'B'b'$,

$$B'b' = \frac{M'B' \; Sin. \; B'M'b'}{Sin. \; M'b'B'} = \frac{a \, Sin. \; \phi' \; Sin. \; \beta'}{Sin. \; \alpha' \; Sin \; \gamma'},$$

in the triangle $C'N'D'$,

$$C'N' = \frac{C'D' \, Sin. \, C'D'N'}{Sin. \; C'N'D'} = \frac{c \; Sin. \; (\lambda' - \phi')}{Sin. \; \delta'}.$$

Now, since $B'b' = C'N' + N'c'$: we therefore have the equation,

$$\frac{a \; Sin. \; \phi' \; Sin. \; \beta'}{Sin. \; \alpha' \; Sin. \; \gamma'} = \frac{c \; Sin. \; (\lambda' - \phi')}{Sin. \; \delta'} + \frac{b \; Sin. \; (\kappa' - \phi')}{Sin. \; \gamma'.},$$

whence, by the usual method, we obtain

$$Cot. \; \phi' = \frac{a Sin. \beta' \, Sin. \delta' + b Sin. \alpha' \, Sin. \delta' \, Cos. \kappa' + c Sin. \alpha' Sin. \gamma' Cos. \lambda'}{b \, Sin. \; \alpha' \; Sin. \; \delta' \; Sin. \; \kappa' + c \; Sin. \; \alpha' \; Sin. \; \gamma' \; Sin. \lambda'},$$

In order to perceive the agreement of this formula with the foregoing, it is merely necessary to express κ, λ by κ', λ'. Now, since $\kappa = \kappa' - 540$, $\lambda = 720^0 - \lambda'$; then $Sin. \; \kappa = Sin. \; (\kappa' - 540^0) = - Sin. \, (540^0 - \kappa') = - Sin. \; (180^0 - \kappa') = - Sin. \; \kappa'$, $Cos. \; \kappa = Cos. \; (\kappa' - 540^0) = Cos. \, (540^0 - \kappa') = Cos. \, (180^0 - \kappa') = - Cos. \; \kappa'$, $Sin. \lambda = Sin. \, (720^0 - \lambda') = Sin. - \lambda' = - Sin. \lambda'$, $Cos. \, (720^0 - \lambda') = Cos. - \lambda') = Cos. \, \lambda'$. If we substitute these values of $Sin. \; \kappa$, $Cos. \; \kappa$, $Sin. \; \lambda$, $Cos. \; \lambda$, in the first formula, we then obtain the second.

SECTION CV.

PROB. *Of the three problems solved in §§ CII, CIII, CIV, that in § CIII is the most general; it includes both the others as single cases. But there are besides some*

other remarkable cases, whose solution may be derived from it without much trouble. We shall now proceed to show how this is done.

First Case.

Let the point P (*fig.* 115) move into the line CN, as *fig.* 121 shows. For this case, in § 103, $\delta=0$, $\varepsilon=180^0$, and $\therefore$ $Sin.\ \delta=0$, $Sin.\ \varepsilon=0$. If we make the substitution in the formula there found for $Cot.\ \phi$, then $Cot.\ \phi=\frac{0}{0}$, which leaves its value undetermined. But we must necessarily obtain an expression of this kind, because the position of the point P in the line CN is not given.

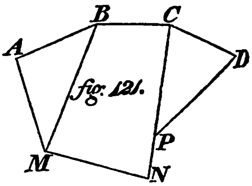


fig. 121.

Divide the numerator and denominator of the above expression by $Sin.\ \delta$; this gives

$$Cot.\ \phi =$$

$$\frac{a\,Sin.\beta Sin.\zeta + b\,Sin.\alpha Sin.\zeta Cos.\kappa + c\,Sin.\alpha Sin.\gamma Cos.\lambda.\frac{Sin.\ \varepsilon}{Sin.\ \delta}}{-b\,Sin.\alpha\,Sin.\zeta\,Sin.\kappa + c\,Sin.\alpha\,Sin.\gamma\,Sin.\lambda.\frac{Sin.\ \varepsilon}{Sin.\ \delta}},$$

Now let (*fig.* 115), $CN=f$, $CP=g$. Then $\frac{Sin.\ \varepsilon}{Sin.\ \delta}=\frac{f}{g}$, and

$$Cot.\ \phi =$$

$$\frac{a\,Sin.\ \beta\ Sin.\ \zeta + b\,Sin.\ \alpha\,Sin.\ \zeta\,Cos.\ \kappa + \frac{cf}{g}\,Sin.\alpha\,Sin.\gamma\,Cos.\lambda}{-b\,Sin.\ \alpha\,Sin.\ \zeta\,Sin.\ \kappa + \frac{cf}{g}Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \lambda};$$

in which

$$\kappa=B+\alpha+\beta-360^0, \lambda=900^0-(\alpha+\beta+\gamma+\delta+\varepsilon+\zeta+B+C).$$

This expression is always correct, wherever the point P may be situated; consequently also when, as in *fig.* 121, it is in CN. But for this case $\delta=0$, $\varepsilon=180^0$, consequently $\lambda = 720^0-(\alpha+\beta+\gamma+\zeta+B+C)$.

Now put (*fig.* 121) $f=g$: then P is situated in N, and the figure is transformed into *fig.* 118. We then have

$$Cot.\ \phi = \frac{a\ Sin.\ \beta\ Sin.\ \zeta + b\ Sin.\ \alpha\ Sin.\ \zeta\ Cos.\ \kappa + c\ Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \lambda}{-b\ Sin.\ \alpha\ Sin.\ \zeta\ Sin.\ \kappa + c\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \lambda},$$

which completely coincides with § CIV, merely by putting, suitably to the notation there used, δ for ζ.

Second Case.

Again, let the point D (*fig.* 118) move into B: then this figure is transformed into *fig.* 112, and ζ into $-\zeta$; further, $C=0$, and $c=b$. Then the expression just found is transformed into the following one:

$$Cot.\ \phi' = \frac{-a\ Sin.\ \beta'\ Sin.\ \zeta' - b\ Sin.\ \alpha\ Sin.\ \zeta\ Cos.\ \kappa' + b\ Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \lambda}{b\ Sin.\ \alpha\ Sin.\ \zeta\ Sin.\ \kappa' + b\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \lambda},$$

where $\kappa' = B+\alpha+\beta-360^0$, $\lambda' = 720^0-(\alpha+\beta+\gamma+\zeta+B)$.

If this expression be compared with that found in § CII for the case, we must, in the first place, to suit the notation there used, put δ for γ, and γ for ζ. Then $\lambda' = 720^0 - (\alpha+\beta+\delta-\gamma+B)$. In the section above-mentioned, $\kappa = B+\alpha+\beta-360^0$, $\lambda = B+\alpha+\beta-\gamma+\delta-360^0$; we $\therefore$ have $\kappa' = \kappa$, $\lambda' = 360^0 - \lambda$; consequently $Sin.\ \kappa' = Sin.\ \kappa$, $Cos.\ \kappa' = Cos.\ \kappa$, $Sin.\ \lambda' = -Sin.\ \lambda$, $Cos.\ \lambda' = Cos.\ \lambda$. The substitution of these values in the above expression, after making the proper change with respect to γ and ζ, gives

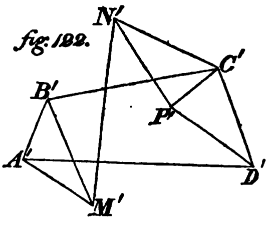
$$Cot.\ \phi = \frac{a\ Sin.\ \beta\ Sin.\ \gamma + b\ Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \kappa - b\ Sin.\ \alpha\ Sin.\ \delta\ Cos.\ \lambda}{-b\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \kappa + b\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \kappa}.$$

which perfectly agrees with § CII.

Third Case.

I shall now assume, that the points A, B, C, D, M, N, P, in *fig.* 115, have the positions A', B', C', D', M', N', P', (*fig.* 122), and that it is required to find for this and all

other similar figures a particular formula by which the position of M', N', P' may be determined. Let the measured angles be $A'M'B' = \alpha'$, $B'M'N' = \beta'$, $C'N'M' = \gamma'$, $C'N'P' = \delta'$, $C'P'N' = \varepsilon'$, $C'P'D' = \zeta'$, the given ones $A'B'C' = B'$, $B'C'D' = C'$. Let the required angle be $M'A'B' = \phi'$.



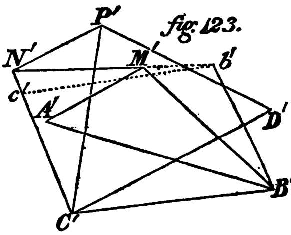
The comparison of *figs.* 115 and 122, in reference to § CIII, gives $B = B'$, $C = C'$, $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = -\gamma'$, $\delta = \delta'$, $\varepsilon = \varepsilon'$, $\zeta = \zeta'$, $\phi = \phi'$; the substitution of these values in the formula there found for *Cot.* ϕ, consequently gives

$$Cot\ \phi' = \frac{\left[\begin{array}{c} a\ Sin.\ \beta'\ Sin.\ \delta'\ Sin.\ \zeta' + b\ Sin.\ \alpha'\ Sin.\ \delta'\ Sin.\ \zeta'\ Cos.\ \kappa' \\ -\ c\ Sin.\ \alpha'\ Sin.\ \gamma'\ Sin.\ \varepsilon'\ Cos.\ \lambda \end{array}\right]}{-b\ Sin.\ \alpha'\ Sin.\ \delta'\ Sin.\ \zeta'\ Sin.\ \kappa' - c\ Sin.\ \alpha'\ Sin.\ \gamma'\ Sin.\ \varepsilon'\ Sin.\ \lambda'},$$

in which $\kappa' = B' + \alpha' + \beta' - 360^0$, $\lambda' = 900^0 - (\alpha' + \beta' - \gamma' + \delta' + \varepsilon' + \zeta' + B' + C')$.

Fourth Case.

Let A', B', C', D', (*fig.* 123) be the four given points, M', N', P', the three whose position is sought. We have the following measured angles: $A'M'B' = \alpha'$, $B'M'N' = \beta'$, $C'N'M' = \gamma'$, $C'N'P' = \delta'$, $C'P'N' = \varepsilon'$, $C'P'D' = \zeta'$. These angles are given, viz. $A'B'C' = B'$, $B'C'D' = C'$, and the sides $A'B' = a$, $B'C = b$, $C'D' = c$, the required angle $M'A'B' = \phi'$. Comparing this with the scheme in *fig.* 115, gives $\alpha = -\alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, $\delta = -\delta'$, $\varepsilon = -\varepsilon'$, $\zeta = -\zeta'$, $B = B'$, $C = C'$, $\phi = -\phi'$. Substituting these values in the formula in § CIII, we get



$$-\text{Cot.}\,\phi' = \frac{\left[\begin{array}{c} a\,\text{Sin.}\,\beta'\,\text{Sin.}\,\delta\,\text{Sin.}\,\zeta' - b\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\delta'\,\text{Sin.}\,\zeta'\,\text{Cos.}\,\kappa' \\ + c\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\gamma'\,\text{Sin.}\,\varepsilon'\,\text{Cos.}\,\lambda' \end{array}\right]}{b\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\delta'\,\text{Sin.}\,\zeta'\,\text{Sin.}\,\kappa' + c\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\gamma'\,\text{Sin.}\,\varepsilon'\,\text{Sin.}\,\lambda'},$$

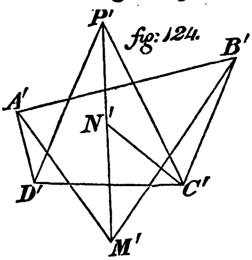
or

$$\text{Cot.}\,\phi' = \frac{\left[\begin{array}{c} -a\,\text{Sin.}\,\beta'\,\text{Sin.}\,\delta'\,\text{Sin.}\,\zeta' + b\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\delta'\,\text{Sin.}\,\zeta'\,\text{Cos.}\,\kappa' \\ - c\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\gamma'\,\text{Sin.}\,\varepsilon'\,\text{Cos.}\,\lambda \end{array}\right]}{b\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\delta'\,\text{Sin.}\,\zeta'\,\text{Sin.}\,\kappa' + c\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\gamma'\,\text{Sin.}\,\varepsilon'\,\text{Sin.}\,\lambda'},$$

in which $\kappa' = B' - \alpha' + \beta' - 360^0$, $\lambda' = 900^0 - (-\alpha' + \beta' + \gamma' - \delta' - \varepsilon' - \zeta' + B' + C')$.

Fifth Case.

Let A', B', C', D', (*fig.* 124) be the four given points, and M', N', P', the three points sought; the latter are here in a direct line. Let $A'B'C' = B'$, $B'C'D' = C'$, $A'M'B' = \alpha'$, $B'M'N' = \beta'$, $C'N'M' = \gamma'$, $C'N'P' = \delta' = 180^0 - \gamma'$, $C'P'N' = \varepsilon'$, $C'P'D' = \zeta'$; the required angle $M'A'B' = \phi'$. The comparison with *fig.* 115 gives $\alpha = \alpha'$, $\beta = -\beta'$, $\gamma = -\gamma'$, $\delta = -\delta'$, $\varepsilon = -\varepsilon'$, $\zeta = \zeta'$, $B = B'$, $C = C'$, $\phi = \phi'$. Consequently the substitution of these values in the formula, § CIII, gives



$$\text{Cot.}\,\phi' = \frac{\left[\begin{array}{c} a\,\text{Sin.}\,\beta'\,\text{Sin.}\,\delta\,\text{Sin.}\,\zeta' - b\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\delta'\,\text{Sin.}\,\zeta'\,\text{Cos.}\,\kappa' \\ + c\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\gamma'\,\text{Sin.}\,\varepsilon'\,\text{Cos.}\,\lambda' \end{array}\right]}{b\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\delta'\,\text{Sin.}\,\zeta'\,\text{Sin.}\,\kappa' + c\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\gamma'\,\text{Sin.}\,\varepsilon'\,\text{Sin.}\,\lambda'},$$

or, since $\text{Sin.}\,\delta' = \text{Sin.}\,\gamma'$,

$$\text{Cot.}\,\phi' = \frac{a\,\text{Sin.}\,\beta'\,\text{Sin.}\,\zeta' - b\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\zeta'\,\text{Cos.}\,\kappa' + c\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\varepsilon'\,\text{Cos.}\,\lambda'}{b\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\zeta'\,\text{Sin.}\,\kappa' + c\,\text{Sin.}\,\alpha'\,\text{Sin.}\,\varepsilon'\,\text{Sin.}\,\lambda'},$$

in which

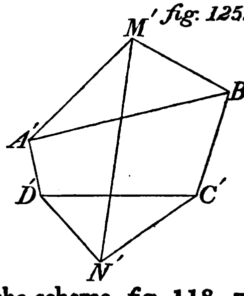
$\kappa' = B' + \alpha' - \beta' - 360^0$, $\lambda' = 900^0 - (\alpha' - \beta' - \gamma' - \delta' - \varepsilon' + \zeta' + B' + C') = 1080^0 - (\alpha' - \beta' - \varepsilon' + \zeta' + B' + C')$, because $\gamma' + \delta' = 180^0$. The angle $M'A'B'$, and consequently the position of the points M', F', depends $\therefore$ for this case not on the angles γ', δ'; which, indeed, is the case.

If the points C', D', are in the line $A'B'$, then the formula here found for *Cot.* ϕ' remains the same; but the letters κ', λ', have different values, according to the order in which the points A', B', C', D', are relatively situated. Thus, if these four points be placed in the following order, viz. A', D', B', C', then $B' = 180^0$, $C' = 0$; consequently $\kappa' = \alpha' - \beta' - 180^0$, $\lambda = 900^0 - (\alpha' - \beta' - \varepsilon' + \zeta')$. If they succeed one another in the order D', A', B', C': then $B' = 180^0$, $C' = 0$, and κ', λ', have the same values. But if they are in the order A', B', C', D': then $B' = 180$, $C' = 180^0$, and $\therefore$ $\kappa' = \alpha' - \beta' - 180^0$, $\lambda' = 540^0 - (\alpha' - \beta' - \varepsilon' + \zeta')$.

Sixth Case.

A', B, C', D', (*fig.* 125), is a given quadrilateral M', N', are two points, whose position is required to be determined. From M', only the side $A'B'$, and the point N' can be seen; from N' only the side $C'D'$, and the point M' are seen. The measured angles are $A'M'B' = \alpha'$, $B'M'N' = \beta'$, $C'N'M' = \gamma'$, $C'N'D' = \delta'$. We have given, $A'B' = a$, $B'C' = b'$, $C'D' = c$, $A'B'C' = B'$, $B'C'D' = C'$. Let the required angle $M'A'B' = \phi$. By comparing with the scheme, *fig.* 118, we get, in reference to § CIV, $B = B'$, $C = C'$, $\alpha = -\alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, $\delta = -\delta'$, $\phi = -\phi'$. By the substitution of these values in the formula there found for *Cot.* ϕ, we obtain,

fig. 125.



$$- Cot.\ \phi' = \frac{-a Sin.\beta' Sin.\delta' + b Sin.\alpha' Sin.\delta' Cos.\kappa' - c Sin.\alpha' Sin.\gamma' Cos.\lambda'}{-b\ Sin.\ \alpha'\ Sin.\ \delta'\ Sin.\ \kappa' - c\ Sin.\ \alpha'\ Sin.\ \gamma'\ Sin.\ \lambda'}$$

or,

$$Cot.\ \phi' = \frac{-a\ Sin.\ \beta'\ Sin.\ \delta' + b\ Sin.\ \alpha'\ Sin.\ \delta'\ Cos.\ \kappa' - c\ Sin.\ \alpha'\ Sin.\ \gamma'\ Cos.\ \lambda'}{b\ Sin.\ \alpha'\ Sin.\ \delta'\ Sin.\ \kappa' + \epsilon\ Sin.\ \alpha'\ Sin.\ \gamma'\ Sin.\ \lambda'}$$

in which

$\kappa' =$
$B' - \alpha' + \beta' - 360^0$, $\lambda' = 720^0 - (-\alpha' + \beta' + \gamma' - \delta' + B' + C')$.

Again let the point D' move into A'. Then $a = 0$, the quadrilateral $A'B'C'D'$ is transformed into a triangle, and we obtain,

$$Cot.\ \phi' = \frac{b\ Sin.\ \delta'\ Cos.\ \kappa' - c\ Sin.\ \gamma'\ Cos.\ \lambda'}{b\ Sin.\ \delta'\ Sin.\ \kappa' - c\ Sin.\ \gamma'\ Sin.\ \lambda'}.$$

SECTION CVI.

We can likewise, if we choose, solve the formula for every similar figure immediately from this figure itself, as was shown in § CIV by an example. As this is a very useful practice for the young geometrician, I will here give another example, and with this view make use of the fourth case in the preceding section.

Draw, (*fig.* 123) $B'b'$ parallel to $C'N'$, which meets $M'N'$ produced in b', and from b' draw $b'c'$ parallel to $B'C'$. We then have, $B'b'M' = 180^0 - C'N'M' = 180^0 - \gamma'$, $M'B'b' = B'M'N' - B'b'M' = \beta' + \gamma' - 180^0$, $A'B'M' = 180^0 - (A'M'B' + M'A'B') = 180^0 - (\alpha' + \phi')$, $C'B'b' = A'B'C' + A'B'M' + M'B'b' = B' + \beta' + \gamma' - \alpha' - \phi' = C'c'b'$, $N'b'c' = C'c'b' - C'N'M' = B' + \beta' - \alpha' - \phi'$; further, $B'C'c = 180^0 - C'c'b' = 180^0 - B' - \beta' - \gamma' + \alpha' + \phi'$, $D'C'N' = B'C'c' - B'C'D' = 180^0 - B' - C' - \beta' - \gamma' + \alpha' + \phi'$; $\therefore$ in the quadrilateral $D'C'N'P$, $C'D'P' = 360^0 - D'C'N' - C'N'P' - N'P'D' = 360^0 - D'C'N' - C'N'P' - C'P'N' - C'P'D' = 180^0 + B' + C' + \beta' + \gamma' - \alpha' - \phi' - \delta' - \epsilon' - \zeta'$. If $\therefore$, for shortness-sake, we put, $B' + \beta' - \alpha' = \kappa''$, $180^0 + B' + C' + \beta' + \gamma' - \alpha'$

$-\delta' - \varepsilon' - \zeta' = \lambda''$: then $N'b'c' = \kappa'' - \phi'$, $C'D'P' = \lambda'' - \phi'$.

Consequently in the triangle $A'B'M'$, we have

$$B'M' = \frac{A'B' \text{ Sin. } M'A'B'}{\text{Sin. } A'M'B'} = \frac{a \text{ Sin. } \phi'}{\text{Sin. } \alpha'},$$

in the triangle $B'b'M'$,

$$B'b' = \frac{B'M' \text{ Sin. } B'M'b}{\text{Sin. } B'b'M'} = \frac{a \text{ Sin. } \phi' \text{ Sin. } \beta'}{\text{Sin. } \alpha' \text{ Sin. } \gamma'},$$

in the triangle $N'b'c'$,

$$N'c' = \frac{b'c' \text{ Sin. } N'b'c'}{\text{Sin. } c'N'b'} = \frac{b \text{ Sin. } (\kappa'' - \phi)}{\text{Sin. } \gamma'},$$

in the triangle $C'D'P'$,

$$C'P' = \frac{C'D' \text{ Sin. } C'D'P'}{\text{Sin. } C'P'D} = \frac{c \text{ Sin. } (\lambda'' - \phi')}{\text{Sin. } \zeta'},$$

in the triangle $C'P'N'$,

$$C'N' = \frac{C'P' \text{ Sin. } C'P'N}{\text{Sin. } C'N'P'} = \frac{c \text{ Sin. } (\lambda'' - \phi') \text{ Sin. } \varepsilon'}{\text{Sin. } \delta' \text{ Sin. } \zeta'},$$

Now $C'N' = C'c' + N'c' = B'b' + N'c'$: we $\therefore$ have the equation

$$\frac{c \text{ Sin. } (\lambda'' - \phi') \text{ Sin. } \varepsilon'}{\text{Sin. } \delta' \text{ Sin. } \gamma'} = \frac{a \text{ Sin. } \phi' \text{ Sin. } \beta'}{\text{Sin. } \alpha' \text{ Sin. } \gamma'} + \frac{b \text{ Sin. } (\kappa'' - \phi')}{\text{Sin. } \gamma'}.$$

whence we obtain,

$$\text{Cot. } \phi' = \frac{\left[\begin{array}{c} a \text{ Sin. } \beta' \text{ Sin. } \delta' \text{ Sin. } \zeta' - b \text{ Sin. } \alpha' \text{ Sin. } \delta' \text{ Sin. } \zeta' \text{ Cos. } \kappa'' \\ + c \text{ Sin. } \alpha' \text{ Sin. } \gamma' \text{ Sin. } \varepsilon' \text{ Cos. } \lambda'' \end{array}\right]}{-b \text{ Sin. } \alpha' \text{ Sin. } \delta' \text{ Sin. } \zeta' \text{ Sin. } \kappa'' + c \text{ Sin. } \alpha' \text{ Sin. } \gamma' \text{ Sin. } \varepsilon' \text{ Sin. } \lambda''}.$$

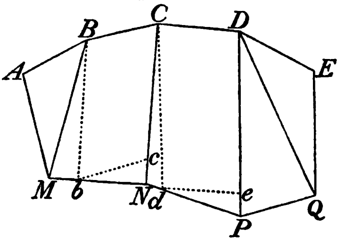
If we compare this formula with that for the fourth case of the preceding section, we find $\kappa'' = 360^\circ + \kappa'$, $\lambda'' = 1080^\circ - \lambda'$; consequently *Sin.* $\kappa'' =$ *Sin.* κ', *Cos.* $\kappa'' =$ *Cos.* κ', *Sin.* $\lambda'' = -$ *Sin.* λ', *Cos.* $\lambda'' =$ *Cos.* λ', whence the equality of both formulæ is evident.

SECTION CVII.

Prob. *Five inaccessible points are given in position and distance: required to determine the position of four other points.*

Solut. Let A, B, C, D, E, (*fig.* 126), be the five points, M, N, P, Q, the four required ones. From M, only A, B, N, can be seen, from N, only C, M, P, from P, only D, N, P, and from Q, only D, E, P. Let the given lines and angles be $AB = a$, $BC = b$, $CD = c$, $DE = d$, $ABC = B$, $BCD = C$, $CDE = D$; the measured angles $AMB = \alpha$, $BMN = \beta$, $CNM = \gamma$, $CNP = \delta$, $DPN = \varepsilon$, $DPQ = \zeta$, $DQP = \eta$, $DQE = \theta$. Further, let the required angle be $MAB = \phi$.

fig. 126.



1. Draw Bb parallel to CN, bc to BC, cd to DP, and de to CD. Then $BbM = CNM = \gamma$, $CdN = DPN = \varepsilon$; consequently $MBb = 180^0 - (BMb + BbM) = 180^0 - (\beta + \gamma)$, $NCd = 180^0 - (\delta + \varepsilon)$. Further, in the quadrilateral $ABbM$, $ABb = 360^0 - (AMb + BbM + MAB) = 360^0 - (\alpha + \beta + \gamma + \phi)$, and in the pentagon $ABCNM$, $BCN = 540^0 - (ABC + AMN + CNM + MAB) = 540^0 - (B + \alpha + \beta + \gamma + \phi)$. We consequently have $bBC = ABC - ABb = B + \alpha + \beta + \gamma + \phi - 360^0 = Ccb$, $dCD = BCD - BCN - NCd = B + C + \alpha + \beta + \gamma + \delta + \varepsilon + \phi - 720^0 = Ded$, and $\therefore$ $Nbc = Ccb - bNC = B + \alpha + \beta + \phi - 360^0$; $Pde = Ded - dPD = B + C + \alpha + \beta + \gamma + \delta + \phi - 720^0$. Further, in the nonagon $ABCDEQPNM$, the angle $DEQ = 1260^0 - (ABC + BCD + CDE + AMN + MNP + NPQ + MAB) = 1260^0 - (B + C + D + \alpha + \beta + \gamma + \delta + \varepsilon + \zeta + \eta +$

$\theta + \phi$). Now, if for the sake of brevity, we put, $B + \alpha + \beta - 360^0 = \kappa$, $B + C + \alpha + \beta + \gamma + \delta - 720^0 = \lambda$, $1260^0 - (B + C + D + \alpha + \beta + \gamma + \delta + \varepsilon + \zeta + \eta + \theta) = \mu$: then $Nbc = \kappa + \phi$, $Pde = \lambda + \phi$, $DEQ = \mu - \phi$.

2. In the triangle ABM,

$$BM = \frac{AB \text{ Sin. } MAB}{\text{Sin. } AMB} = \frac{a \text{ Sin. } \phi}{\text{Sin. } \alpha};$$

in the triangle MBb,

$$Bb = \frac{BM \text{ Sin. } BMb}{\text{Sin. } BbM} = \frac{a \text{ Sin. } \phi \text{ Sin. } \beta}{\text{Sin. } \alpha \text{ Sin. } \gamma};$$

in the triangle Nbc,

$$Nc = \frac{bc \text{ Sin. } Nbc}{\text{Sin. } bNc} = \frac{b \text{ Sin. } (\kappa + \phi)}{\text{Sin. } \gamma};$$

consequently $Cn = Cc + Nc = Bb + Nc =$

$$\frac{a \text{ Sin. } \phi \text{ Sin. } \beta}{\text{Sin. } \alpha \text{ Sin. } \gamma} + \frac{b \text{ Sin. } (\kappa + \phi)}{\text{Sin. } \gamma}.$$

Further, in the triangle NCd,

$$Cd = \frac{CN \text{ Sin. } CNd}{\text{Sin. } CdN} = \frac{CN \text{ Sin. } \delta}{\text{Sin. } \varepsilon},$$

and in the triangle Pde,

$$Pe = \frac{de \text{ Sin. } Pde}{\text{Sin. } dPe} = \frac{c \text{ Sin. } (\lambda + \phi)}{\text{Sin. } \varepsilon};$$

consequently $DP = De + Pe = Cd + Pe =$

$$\frac{CN \text{ Sin. } \delta}{\text{Sin. } \varepsilon} + \frac{c \text{ Sin. } (\lambda + \phi)}{\text{Sin. } \varepsilon} =$$

$$\frac{a \text{ Sin. } \phi \text{ Sin. } \beta \text{ Sin. } \delta}{\text{Sin. } \alpha \text{ Sin. } \gamma \text{ Sin. } \varepsilon} + \frac{b \text{ Sin. } (\kappa + \phi) \text{ Sin. } \delta}{\text{Sin. } \gamma \text{ Sin. } \varepsilon} + \frac{c \text{ Sin. } (\lambda + \phi)}{\text{Sin. } \varepsilon}.$$

3. But an expression may be found for DP. Thus in the triangle DEQ,

$$DQ = \frac{DE \text{ Sin. } DEQ}{\text{Sin. } DQE} = \frac{d \text{ Sin. } (\mu - \phi)}{\text{Sin. } \theta},$$

and in the triangle DPQ,

$$DP = \frac{DQ\ Sin.\ DQP}{Sin.\ DPQ} = \frac{d\ Sin.\ (\mu - \phi)\ Sin.\ \eta}{Sin.\ \theta\ Sin.\ \zeta}.$$

4. If we equate the expressions found for DP in 2 and 3, we then obtain the equation,

$$\frac{a\ Sin.\ \phi\ Sin.\ \beta\ Sin.\ \delta}{Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \varepsilon} + \frac{b\ Sin.\ (\kappa + \phi)\ Sin.\ \delta}{Sin.\ \gamma\ Sin.\ \varepsilon},$$

$$+ \frac{c\ Sin.\ (\lambda + \phi)}{Sin.\ \varepsilon} = \frac{d\ Sin.\ (\mu - \phi)\ Sin.\ \eta}{Sin.\ \theta\ Sin.\ \zeta},$$

or, when we solve $Sin.\ (\kappa + \phi)$, $Sin.\ (\lambda + \phi)$, $Sin.\ (\mu - \phi)$ divide by, $Sin.\ \phi$, and substitute $Cot.\ \phi$ for $\frac{Cos.\ \phi}{Sin.\ \phi}$,

$$\frac{a\ Sin.\ \beta\ Sin.\ \delta}{Sin.\ a\ Sin.\ \gamma\ Sin.\ \varepsilon} + \frac{b\ Sin.\ \delta\ (Sin.\ \kappa\ Cot.\ \phi + Cos.\ \kappa)}{Sin.\ \gamma\ Sin.\ \varepsilon} +$$

$$\frac{c\ (Sin.\ \lambda\ Cot.\ \phi + Cos.\ \lambda)}{Sin.\ \varepsilon} = \frac{d\ Sin.\ \eta\ (Sin.\ \mu\ Cot.\ \phi - Cos.\ \mu)}{Sin.\ \theta\ Sin.\ \zeta}.$$

Hence we obtain,

$$Cot.\ \phi = \frac{\left[\begin{array}{c} a\ Sin.\beta\ Sin.\delta\ Sin.\zeta\ Sin.\theta + b\ Sin\ \alpha\ Sin.\delta\ Sin.\zeta\ Sin.\theta\ Cos.\kappa + \\ c\ Sin.\alpha\ Sin.\gamma\ Sin.\zeta\ Sin.\theta\ Cos.\lambda + d\ Sin.\alpha\ Sin.\gamma\ Sin.\varepsilon\ Sin.\eta\ Cos.\mu \end{array}\right]}{\left[\begin{array}{c} -b\ Sin.\alpha\ Sin.\delta\ Sin.\zeta\ Sin.\theta\ Sin.\kappa - c\ Sin.\alpha\ Sin\ \gamma\ Sin.\zeta\ Sin.\theta\ Sin.\lambda \\ + d\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \varepsilon\ Sin.\ \eta\ Sin.\ \mu \end{array}\right]}$$

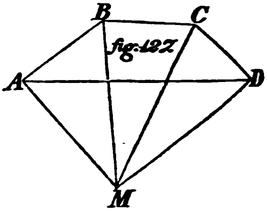
REMARK. In a similar manner we can generally solve the problem when n points are given, to determine the position of $n - 1$ other points.

SECTION CVIII.

PROB. *In a quadrilateral, two of its opposite sides, together with the angles which they make with one of the other two sides, are given. This quadrilateral is seen from any one position, and from hence the apparent distances of the angular points from one another*

are measured; find from these data the other two sides, and the distance of the position from each angular point.

SOLUT. Let $ABCD$ (*fig.* 127) be the quadrilateral, M the position, the given sides $AB = a$, $CD = b$, and angles $ABC = B$, $BCD = C$. The measured angles $ABM = \alpha$, $BMC = \beta$, $CMD = \gamma$.



1. If the angle MAB is known, we then likewise have all the rest of the figure. Assume $\therefore$ $MAB = \phi$. Then $ABM = 180^0 - (\alpha + \phi)$, $MBC = ABC - ABM = B + \alpha + \phi - 180^0$, $BCM = 180^0 - (MBC + BMC) = 360^0 - (B + \alpha + \beta + \gamma + \phi)$. Further, in the pentagon $MABCD$, $MDC = 540^0 - (B + C + \alpha + \beta + \gamma + \phi)$. For the sake of brevity, put $B + \alpha - 180^0 = \kappa$, $360^0 - (B + \alpha + \beta) = \lambda$, $540^0 - (B + C + \alpha + \beta + \gamma) = \mu$; this gives, $MBC = \kappa + \phi$, $BCM = \lambda - \phi$, $MDC = \mu - \phi$,

2. In the triangle ABM,

$$BM = \frac{AB\ Sin.\ MAB}{Sin.\ AMB} = \frac{a\ Sin.\ \phi}{Sin.\ \alpha},$$

in the triangle BMC,

$$MC = \frac{BM\ Sin.\ MBC}{Sin.\ BCM} = \frac{a\ Sin.\ \phi\ Sin.\ (\kappa + \phi)}{Sin.\ \alpha\ Sin.\ (\lambda - \phi)}.$$

Further, in the triangle MCD,

$$MC = \frac{CD\ Sin.\ MDC}{Sin.\ CMD} = \frac{b\ Sin.\ (\mu - \phi)}{Sin.\ \gamma}.$$

3. If the two values found for MC are equated, we then obtain,

$$\frac{a\ Sin.\ \phi\ Sin.\ (\kappa + \phi)}{Sin.\ \alpha\ Sin.\ (\lambda - \phi)} = \frac{b\ Sin.\ (\mu - \phi)}{Sin.\ \gamma},$$

or,

$$a\ Sin.\ \gamma\ Sin.\ (\kappa + \phi) = b\ Sin.\ \alpha\ Sin.\ (\lambda - \phi)\ Sin.\ (\mu - \phi).$$

4. Now $Sin.\,\phi\ Sin.\,(\kappa+\phi) = \frac{1}{2}[Cos.\,\kappa - Cos.\,(\kappa+2\phi)]$, $Sin.(\lambda-\phi)\,Sin.\,(\mu-\phi)=\frac{1}{2}[Cos.(\lambda-\mu)-Cos.(\lambda+\mu-2\phi)]$; we consequently have

$$a\ Sin.\,\alpha[Cos.\,(\lambda+\mu-2\phi)]=$$
$$b\ Sin.\,\alpha\,[Cos.\,(\lambda-\mu) - Cos.\,(\lambda+\mu-2\phi)].$$

or,

$$a\ Sin.\,\gamma\ Cos.\,\kappa - b\ Sin.\,\alpha\ Cos.\,(\lambda-\mu) =$$
$$a\ Sin.\,\gamma\ Cos.\,(\kappa+2\phi) - b\ Sin.\,\alpha\,Cos.\,(\lambda+\mu-2\phi).$$

Now since $Cos.\,(\kappa+2\phi) = Cos.\,\kappa\ Cos.\,2\phi - Sin.\,\kappa\,Sin.\,2\phi$, $Cos.\,(\lambda+\mu-2\phi) = Cos.\,(\lambda+\mu)\ Cos.\,2\phi + Sin.\,(\lambda+\mu)$ $Sin.\,2\phi$: therefore

$$a\ Sin.\,\gamma\ Cos.\,\kappa - b\ Sin.\,\alpha\ Cos.\,(\lambda-\mu) =$$
$$[a\ Sin.\,\gamma\ Cos.\,\kappa - b\ Sin.\,\alpha\ Cos.\,(\lambda+\mu)]\ Cos.\,2\phi$$
$$-[a\ Sin.\,\gamma\ Sin.\,\kappa + b\ Sin.\,\alpha\ Sin.\,(\lambda+\mu)]\ Sin.\,2\phi.$$

5. Divide this equation by $a\ Sin.\,\gamma\ Sin.\,\kappa + b\ Sin.\,\alpha$ $Sin.\,(\lambda+\mu)$, and put

$$\frac{a\ Sin.\,\gamma\ Cos.\,\kappa - b\ Sin.\,\alpha\ Cos.\,(\lambda+\mu)}{a\ Sin.\,\gamma\ Sin.\,\kappa + b\ Sin.\,\alpha\ Sin.\,(\lambda+\mu)} = Tan.\,\omega;$$

this gives

$$Tan.\,\omega\ Cos.\,2\phi - Sin.\,2\phi =$$
$$\frac{a\ Sin.\,\gamma\ Cos.\,\kappa - b\ Sin.\,\alpha\ Cos.\,(\lambda-\mu)}{a\ Sin.\,\gamma\ Sin.\,\kappa + b\ Sin.\,\alpha\ Sin.\,(\lambda+\mu)},$$

or, since $Tan.\,\omega\ Cos.\,2\phi - Sin.\,2\phi =$

$$\frac{Sin.\,\omega\ Cos.\,2\phi - Cos.\,\omega\ Sin.\,2\phi}{Cos.\,\omega} = \frac{Sin.\,(\omega-2\phi)}{Cos.\,\omega},$$

$$Sin.\,(\omega-2\phi) =$$
$$\frac{a\ Sin.\,\gamma\ Cos.\,\kappa\ Cos.\,\omega - b\ Sin.\,\alpha\ Cos.\,(\lambda-\mu)\ Cos.\,\omega}{a\ Sin.\,\gamma\ Sin.\,\kappa + b\ Sin.\,\alpha\ Sin.\,(\lambda+\mu)};$$

whence ϕ may be determined.

Exam. Let $a=360$, $b=300$, $B=140^\circ.\ 26'$, $C=136^\circ.\ 22'$, $\alpha=36^\circ.\ 20'$, $\beta=29^\circ.\ 41'$, $\gamma=27^\circ.\ 38'$. Here $\kappa=-3^\circ.\ 14'$, $\lambda=153^\circ.\ 33'$, $\mu=169^\circ.\ 33'$.

$$a \; Sin. \gamma \; Cos. \kappa \quad = \quad 166 \cdot 7063$$
$$b \; Sin. \alpha \; Cos. (\lambda + \mu) = \quad 142 \cdot 1396$$
$$a \; Sin. \gamma \; Sin. \kappa \quad = - \quad 9 \cdot 4176$$
$$b \; Sin. \alpha \; Sin. (\lambda + \mu) = - 106 \cdot 7214 ;$$

consequently,

$$Tan. \omega = \frac{166 \cdot 7063 - 142 \cdot 1396}{-9 \cdot 4176 - 106 \cdot 7214} = - 0 \cdot 2115284$$

and $\therefore$

$$\omega = 168^0. \; 3'. \; 23''.$$

We consequently have,

$$a \; Sin. \gamma \; Cos. \kappa \; Cos. \omega \quad = - 163 \cdot 0974$$
$$b \; Sin. \alpha \; Cos. (\lambda - \mu) \; Cos. \omega = - 167 \cdot 1602,$$

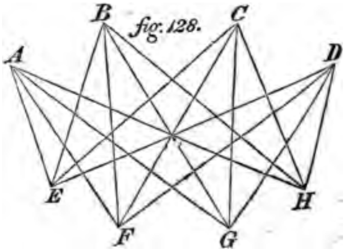
and $\therefore$

$$Sin. (\omega - 2\phi) = \frac{-163 \cdot 0974 + 167 \cdot 1602}{-9 \cdot 4176 - 106 \cdot 7214} = - 0 \cdot 0349822$$
$$\omega - 2\phi = - 2^0. \; 0'. \; 10''.$$
$$\phi = 85^0. \; 1'. \; 46''.$$

SECTION CIX.

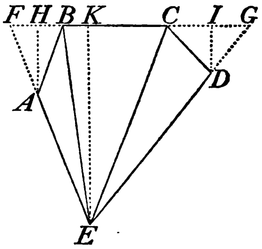
PROB. *Four objects are seen from four stations, and at each of these stations the apparent distances of the objects from one another are measured: from hence determine the position of all the eight points.*

SOLUT. the four objects A, B, C, D (*fig.* 128) are seen from the four stations E, F, G, H, and there the angles $AEB = \alpha$, $BEC = \beta$, $CED = \gamma$, $AFB = \alpha'$, $BFC = \beta'$, $CFD = \gamma'$, $AGB = \alpha''$, $BGC = \beta''$, $CGD = \gamma''$, $AHB = \alpha'''$, $BHC = \beta'''$, $CHD = \gamma'''$, are measured.



1. Join the points A, B, C, D, (*fig.* 129) by the lines AB, BC, CD; find first an equation for the point E. With this view, produce the lines BC, EA, ED, till they meet in F, G, and draw the perpendiculars AH, DI, EK. Since angles only are here given, consequently we cannot determine the actual magnitude of the lines belonging to the figure, but merely their proportion to one another. We can ∴ assume one of these lines, say BC, for unity, and put $BC = 1$. Now if besides the lines AB, CD, and the angles ABC, BCD, or the adjacent angles ABF, DCG, are known; then the position of the four points A, B, C, D, and consequently also the situation of the points E, F, G, H, are determined. Assume ∴ $AB = x$, $CD = y$, $ABF = \phi$, $DCG = \psi$, and moreover, for shortness' sake, $BAE = A$, $CDE = D$, which last two angles vanish further on in the calculation. We then have $AFB = BAE - ABF = A - \phi$, $CGD = CDE - DCG = D - \psi$, $CBE = AFB + AEB = A - \phi + \alpha$, $BCE = CGD + CED = D - \psi + \gamma$.

fig. 129.



2. In the triangle ABE,

$$BE = \frac{AB \text{ Sin. } BAE}{\text{Sin. } AEB} = \frac{x \text{ Sin. } A}{\text{Sin. } \alpha},$$

and in the triangle CED,

$$CE = \frac{CD \text{ Sin. } CDE}{\text{Sin. } CED} = \frac{y \text{ Sin. } D}{\text{Sin. } \gamma};$$

consequently in the right-angled triangle BEK,

$$BK = BE \text{ Cos. } CBE = \frac{y \text{ Sin. } A \text{ Cos. } (A - \phi + \alpha)}{\text{Sin. } \gamma}.$$

and in the right-angled triangle CEK,

$$CK = CE \text{ Cos. } BCE = \frac{y \text{ Sin. } D \text{ Cos. } (D - \psi + \gamma)}{\text{Sin. } \gamma}.$$

Now since $BK + CK = BC = 1$: we then have the equation

$$\frac{x \, Sin.\, A \, Cos.\, (A - \phi + \alpha)}{Sin.\, \alpha} + \frac{y \, Sin.\, D \, Cos.\, (D - \psi + \gamma)}{Sin.\, \gamma} = 1,$$

or,

$$x \, Sin.\, \gamma \, Sin.\, A \, Cos.\, (A - \phi + \alpha) + y \, Sin.\, \alpha \, Sin.\, D \times Cos.\, (D - \psi + \gamma) = Sin.\, \alpha \, Sin.\, \gamma.$$

3. But $Cos.\, (A - \phi + \alpha) = Cos.\, [A - (\phi - \alpha)] = Cos.\, A \, Cos.\, (\phi - \alpha) + Sin.\, A \, Sin.\, (\phi - \alpha)$, $Cos.\, (D - \psi + \gamma) = Cos.\, [D - (\psi - \gamma)] = Cos.\, D \, Cos.\, (\psi - \gamma) + Sin.\, D \times Sin.\, (\psi - \gamma)$. Substitute these values in the obtained equation, and at the same time put $1 - Cos.\, ^2 A$ for $Sin.\, ^2 A$, $1 - Cos.\, ^2 D$ for $Sin.\, ^2 D$; this gives, $Sin.\, \alpha \, Sin.\, \gamma =$

$$x \, Sin.\, \gamma \, Cos.\, A \, [Sin.\, A \, Cos.\, (\phi - \alpha) - Cos.\, A \, Sin.\, (\phi - \alpha)] + x \, Sin.\, \gamma \, Sin.\, (\phi - \alpha) + y \, Sin.\, \alpha \, Cos.\, D \, [Sin.\, D \, Cos.\, (\psi - \gamma) - Cos.\, D \, Sin.\, (\psi - \gamma)] + y \, Sin.\, \alpha \, Sin.\, (\psi - \gamma)$$

$$= x \, Sin.\, \gamma \, Cos.\, A \, Sin.\, (A - \phi + \alpha) + x \, Sin.\, \gamma \, Sin.\, (\phi - \alpha) + y \, Sin.\, \alpha \, Cos.\, D \, Sin.\, (D - \psi + \gamma) + y \, Sin.\, \alpha \, Sin.\, (\psi - \gamma)$$

$$= x \, Sin.\, \gamma \, Cos.\, A \, Sin.\, CBE + x \, Sin.\, \gamma \, Sin.\, (\phi - \alpha) + y \, Sin.\, \alpha \, Cos.\, D \, Sin.\, BCE + y \, Sin.\, \alpha \, Sin.\, (\psi - \gamma).$$

4. In the triangle BEC we have

$$BC = 1 = \frac{BE \, Sin.\, \beta}{Sin.\, BCE} = \frac{x \, Sin.\, A \, Sin.\, \beta}{Sin.\, \alpha \, Sin.\, BCE},$$

and $\therefore$

$$Sin.\, BCE = \frac{x \, Sin.\, A \, Sin.\, \beta}{Sin.\, \alpha}.$$

In like manner

$$BC = 1 = \frac{CE \, Sin.\, \beta}{Sin.\, CBE} = \frac{y \, Sin.\, D \, Sin.\, \beta}{Sin.\, \gamma \, Sin.\, CBE},$$

and $\therefore$

$$Sin.\, CBE = \frac{y \, Sin.\, D \, Sin.\, \beta}{Sin.\, \gamma}.$$

Substitute these values of $Sin.\ BCE$, $Sin.\ CBE$ in the equation in 3; this gives $Sin.\ \alpha\ Sin.\ \gamma =$

$$xy\ Sin.\ \beta\ Sin.\ D\ Cos.\ A + x\ Sin.\ \gamma\ Sin.\ (\phi - \alpha) + xy\ Sin.\ \beta\ Sin.\ A\ Cos.\ D + y\ Sin.\ \alpha\ Sin.\ (\psi - \gamma)$$

$$= xy\ Sin.\ \beta\ Sin.\ (A + D) + x\ Sin.\ \gamma\ Sin.\ (\phi - \alpha) + y\ Sin.\ \alpha\ Sin.\ (\psi - \gamma)$$

or, since $A + D = 540^0 - (AED + ABC + BCD) = 180^0 - (E - \phi - \psi)$, when, for the sake of brevity, we put $AED = E$,

$$Sin.\ \alpha\ Sin.\ \gamma = xy\ Sin.\ \beta\ Sin.\ [E - (\phi + \psi)] + x\ Sin.\ \gamma\ Sin.\ (\phi - \alpha) + y\ Sin.\ \alpha\ Sin.\ (\psi - \gamma).$$

5. In order to separate the unknown magnitudes from the known, solve this equation; this gives

$$Sin.\ \alpha\ Sin.\ \gamma = xy\ Sin.\ \beta\ [Sin.\ E\ Cos.\ (\phi + \psi) - Cos.\ E \times Sin.\ (\phi + \psi)] + x\ Sin.\ \gamma\ (Sin.\ \phi\ Cos.\ \alpha - Cos.\ \phi Sin.\ \alpha) + y\ Sin.\ \alpha\ (Sin.\ \psi\ Cos.\ \gamma - Cos.\ \psi\ Sin.\ \gamma)$$

$$= xy\ Sin.\ \beta\ [Sin.\ E\ (Cos.\ \phi\ Cos.\ \psi - Sin.\ \phi\ Sin.\ \psi) - Cos.\ E\ (Sin.\ \phi\ Cos.\ \psi + Cos.\ \phi\ Sin.\ \psi)] + x\ Sin.\ \gamma\ (Sin.\ \phi\ Cos.\ \alpha - Cos.\ \phi\ Sin.\ \alpha) + y\ Sin.\ \alpha\ (Sin.\ \psi\ Cos.\ \gamma - Cos.\ \psi\ Sin.\ \gamma).$$

6. If we put $x\ Sin.\ \phi = p$, $x\ Cos.\ \phi = q$, $y\ Sin.\ \psi = r$, $y\ Cos.\ \psi = s$; then this equation is transformed into the following one:

$$(qs - pr) Sin.\ \beta\ Sin.\ E - (ps + qr)\ Sin.\ \beta\ Cos.\ E + p\ Sin.\ \gamma\ Cos.\ \alpha + r\ Sin.\ \alpha\ Cos.\ \gamma - (q + s + 1)\ Sin.\ \alpha\ Sin.\ \gamma = 0,$$

or

$$(qs - pr)\frac{Sin.\ \beta\ Sin.\ E}{Sin.\ \alpha\ Sin.\ \gamma} - (ps + qr)\frac{Sin.\ \beta\ Cos.\ E}{Sin.\ \alpha\ Sin.\ \gamma} + p\ Cot.\ \alpha + r\ Cot.\ \gamma - (q + s + 1) = 0,$$

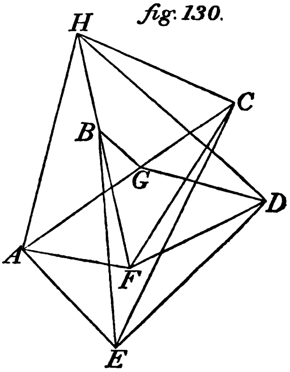
where $E = \alpha + \beta + \gamma$.

Likewise each of the other stations F, G, H gives a similar equation. We ∴ have four equations in all, whence

the value of $qs - pr$, $ps + qr$, p, r, $q + s + 1$, and hence again the values of p, q, r, s, may be determined. If these last are found, then we also have the values of x, y, ϕ, ψ. A complete solution of the above four literal equations is given by Professor Pfleiderer in the Archives for Pure and Practical Mathematics, 10th Number, p. 190. In practice, however, it is much easier to calculate with the given numbers themselves; how this is done will be seen by the following example.

EXAM. I shall assume, that the points E, F, G, H have the position in *fig.* 130. Let $\alpha = AEB = 39^0.\ 38'$, $\beta = BEC = 30^0.\ 54'$, $\gamma = CED = 23^0.\ 21'$, $\alpha' = AFB = 67^0.\ 23'$, $\beta' = BFC = 44^0.\ 44'$, $\gamma' = CFD = 28^0.\ 54'$, $\alpha'' = AGB = 75^0.\ 48'$, $\beta'' = BGC = 105^0.51'$, $\gamma'' = CGD = 50^0.23'$, $\alpha''' = -AHB = -25^0.\ 47'$, $\beta''' = -BHC = -55^0.\ 2'$, $\gamma''' = CHD = 15^0.\ 30'$. Here $E = \alpha + \beta + \gamma = 93^0.\ 53'$, $F = \alpha' + \beta' + \gamma' = 141^0.\ 1'$, $G = \alpha'' + \beta'' + \gamma'' = 232^0.\ 2'$, $H = \alpha''' + \beta''' + \gamma''' = -65^0.\ 19'$. Therefore

fig. 130.



$$\frac{Sin.\ \beta\ Sin.\ E}{Sin.\ \alpha\ Sin.\ \gamma} = 2{\cdot}0266000$$

$$\frac{Sin.\ \beta\ Cos.\ E}{Sin.\ \alpha\ Sin.\ \gamma} = -0{\cdot}1375675$$

$$Cot.\ \alpha = 1{\cdot}2073615$$

$$Cot.\ \gamma = 2{\cdot}3164076$$

$$\frac{Sin.\ \beta'\ Sin.\ F}{Sin.\ \alpha'\ Sin.\ \gamma'} = 0{\cdot}9924784$$

$$\frac{Sin.\ \beta'\ Cos.\ F}{Sin.\ \alpha'\ Sin.\ \gamma'} = -1{\cdot}2263381$$

$$Cot.\ \alpha' = 0{\cdot}4166012$$

$$Cot.\ \gamma' = 1{\cdot}8114969$$

$$\frac{Sin.\ \beta'' \ Sin.\ G}{Sin.\ \alpha' \ Sin.\ \gamma''} = -1{\cdot}0155390$$

$$\frac{Sin.\ \beta'' \ Sin.\ G}{Sin.\ \alpha'' \ Sin.\ \gamma''} = -0{\cdot}7924751$$

$$Cot.\ \alpha'' = 0{\cdot}2530389$$

$$Cot.\ \gamma'' = 0{\cdot}8277620$$

$$\frac{Sin.\ \beta''' \ Sin.\ H}{Sin.\ \alpha''' \ Sin.\ \gamma'''} = -6{\cdot}4057632$$

$$\frac{Sin.\ \beta''' \ Cos.\ H}{Sin.\ \alpha''' \ Sin.\ \gamma'''} = 2{\cdot}9440655$$

$$Cot.\ \alpha''' = -2.0701359$$

$$Cot.\ \gamma''' = 3{\cdot}6058835.$$

We consequently have the four following equations:

$$2{\cdot}0266000\,(qs-pr) + 0{\cdot}1375675\,(ps+qr) + 1{\cdot}2073615\,p + 0{\cdot}3164076\,r - (q+s+1) = 0$$

$$0{\cdot}99924784\,(qs-pr) + 1{\cdot}2263381\,(ps+qr) + 0{\cdot}4166012\,p + 1{\cdot}8114969\,r - (q+s+1) = 0$$

$$-1{\cdot}0155390\,(qs-pr) + 0{\cdot}7924751\,(ps+qr) + 0{\cdot}2530389\,p + 0{\cdot}8277620\,r - (q+s+1) = 0$$

$$-6{\cdot}4057632\,(qs-pr) - 2{\cdot}9440655\,(ps+qr) - 2{\cdot}0701359\,p + 3{\cdot}6058835\,r - (q+s+1) = 0.$$

If the second, third, and fourth be subtracted from the first, we then obtain

$$1{\cdot}0341216\,(qs-pr) - 1{\cdot}0887706\,(ps+qr) + 0{\cdot}7907603\,p + 0{\cdot}5049107\,r = 0.$$

$$3{\cdot}0421390\,(qs-pr) - 0{\cdot}6549076\,(ps+qr) + 0{\cdot}9543226\,p + 1{\cdot}4886456\,r = 0.$$

$$8{\cdot}4323632\,(qs-pr) + 3{\cdot}0816330\,(ps+qr) + 3{\cdot}2774974\,p - 1{\cdot}2894759\,r = 0.$$

Divide the coefficients of each equation by the coefficient of the first term; this gives

$$(qs-pr) - 1{\cdot}0528458\,(ps+qr) + 0{\cdot}7646686\,p + 0{\cdot}4882508\,r = 0$$

$$(qs - pr) - 0{\cdot}2152786\,(ps + qr) + 0{\cdot}3137012\,p + 0{\cdot}4893417\,r = 0$$

$$(qs - pr) + 0{\cdot}3654531\,(ps + qr) + 0{\cdot}3886808\,p - 0{\cdot}1529199\,r = 0$$

and when the two first equations are subtracted from the last,

$$1{\cdot}4182989\,(ps + qr) - 0{\cdot}3759878\,p - 0{\cdot}6411707\,r = 0$$
$$0{\cdot}5807317\,(ps + qr) + 0{\cdot}0749796\,p - 0{\cdot}6422616\,r = 0.$$

Divide again each equation by the coefficient of the first term; this gives

$$(ps + qr) - 0{\cdot}2650977\,p - 0{\cdot}4520702\,r = 0$$
$$(ps + qr) + 0{\cdot}1291123\,p - 1{\cdot}1059524\,r = 0,$$

and when these equations are subtracted from one another,

$$0{\cdot}3942100\,p - 0{\cdot}6538822\,r = 0$$

By means of this equation we successively get the four following equations:

$$\begin{aligned} r &= 0{\cdot}6028762\,p \\ ps + qr &= 0{\cdot}5376401\,p \\ qs - pr &= -0{\cdot}4929713\,p \\ q + s + 1 &= 1{\cdot}6787747\,p \end{aligned}$$

From the first and second equations we obtain

$$s + 0{\cdot}6028762\,q = 0{\cdot}5376401.$$

This equation combined with the fourth, gives

$$\begin{aligned} q &= 4{\cdot}2273334\,p - 3{\cdot}8719414 \\ s &= -2{\cdot}5485587\,p + 2{\cdot}8719414. \end{aligned}$$

If we now substitute the values of q, r, s, in the third equation, we then obtain

$$11{\cdot}3764835\,p^2 - 22{\cdot}5014950\,p + 11{\cdot}1199888 = 0$$

or,

$$p^2 - 1{\cdot}9778955\,p + 0{\cdot}977450 = 0;$$

and this equation gives

$$p = 0{\cdot}9889477 \pm 0{\cdot}0237403,$$

consequently either $p = 1{\cdot}0126880$, or $p = 0{\cdot}9652074$. The two last figures of these values are uncertain.

If we take the first value of p, we find

$$p = 1\cdot0126880 = x \; Sin. \; \phi$$
$$q = 0\cdot4090284 = x \; Cos. \; \phi$$
$$r = 0\cdot6105255 = y \; Sin. \; \psi$$
$$s = 0\cdot2910466 = y \; Cos. \; \psi.$$

From the first and second equations, we obtain

$$Tan. \; \phi = \frac{1\cdot0126880}{0\cdot4090284} = 2\cdot4758378,$$

and from the third and fourth,

$$Tan. \; \psi = \frac{0\cdot6105255}{0\cdot2910466} = 2\cdot0976898\,;$$

consequently

$$\phi = 68^0.\ 0'.\ 21''\cdot7,\ \psi = 64^0.\ 30'.\ 43''\cdot8,$$

and

$$x = \frac{p}{Sin. \; \phi} = 1\cdot09217,\ y = \frac{r}{Sin. \; \psi} = 0\ 67635.$$

If we take the second value of p, we have

$$p = 0\cdot9652074 = x \; Sin. \; \phi$$
$$q = 0\cdot2083121 = x \; Cos. \; \phi$$
$$r = 0\cdot5819006 = y \; Sin. \; \psi$$
$$s = 0\cdot4120537 = y \; Cos. \; \psi$$

and $\therefore$

$$Tan. \; \phi = \frac{0\cdot9652074}{0\cdot2083121} = 4\cdot6334677$$

$$Tan. \; \psi = \frac{0\cdot5819006}{0\cdot4120537} = 1\cdot4121960$$

$$\phi = 77^0.\ 49'.\ 16''\cdot1,\ \psi = 54^0.\ 41'.\ 49''\cdot3$$

$$x = \frac{p}{Sin. \; \phi} = 0\cdot98743, y = \frac{r}{Sin. \; \psi} = 0\cdot71301.$$

Consequently in each of these problems, two different positions of the eight points are possible. Which is the right one, must $\therefore$ be determined from other circumstances.

When the situations of the four points *A*, *B*, *C*, *D*, are determined by means of the angles and lines ϕ, ψ, x, y, when calculated, it is easy, by the problem in § LIV, to determine the position of the points *E*, *F*, *G*, *H*.

REMARK. The problem here solved, incontestably the most important in practical geometry, was invented by the celebrated Lambert; it may be seen in his Contributions II. p. 186, &c. In the calculation of his example, he has only made use of four decimal places; ∴ the expressions there found for P, Q, R, S, or, according to my notation, for p, q, r, s, are not in one case correct as far as this number of decimal places. Thus, according to Lambert, p. 193, $P = 0{\cdot}5690$, $Q = -0{\cdot}3443$ ($-0{\cdot}3543$ is an error of the press), $R = -0{\cdot}4998$, $S = -0{\cdot}3028$; but more properly, $P = 0{\cdot}5698$, $Q = -0{\cdot}3447$, $R = -0{\cdot}5005$, $S = -0{\cdot}3024$. For the angles ϕ, ψ (according to the notation used here), we should only, from these data, have found the degrees correctly, which, in determining the other parts of the figure, would cause great mistakes. It will be advisable, ∴, to perform the calculation throughout with six decimal places.

To this section also belongs the first treatise of the excellent work of Mr. Hauptmann, by Hügenin (Mathematical Contributions for the Formation of the young Geometrician, Königsberg, 1803), which the reader will certainly peruse with advantage and pleasure.

X. ON MAXIMA AND MINIMA, AS FAR AS THIS SUBJECT BELONGS TO ELEMENTARY GEOMETRY.

SECTION CX.

DEFINITION.

A magnitude is said to be a maximum, when it is the greatest of all those which are similar to it; a minimum, when it is the least.

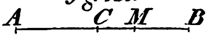
The diameter of a circle is the greatest of all the lines which can be drawn from one point in the circumference of a circle to another, and consequently, in reference to these lines it is a maximum. Further, of all the lines which can be drawn from a given point to a given line, the perpendicular is the least, and consequently, in reference to these lines, it is a minimum.

SECTION CXI.

Prob. *To divide a given line, so that the rectangle contained by the two parts may be a maximum.*

Solut. Let the given line AB (*fig.* 131) be divided in M, so that $AM \times MB$ is greater than any other rectangle which can be contained by any other two parts of this line.

fig. 131.



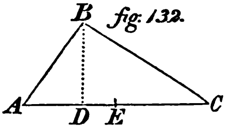
Bisect the line AB in C, and assume $AC = CB = \frac{1}{2} AB = a$, $CM = x$. Then $AM = a + x$, $BM = a - x$, and $AM \times MB = a^2 - x^2$. But the expression $a^2 - x^2$ is evidently the greatest, when $x = 0$; consequently the point M falls in C, and $\therefore$ $AC \times CB$ is the greatest rectangle.

Cor. When ∴ two lines, viz. P, Q, have a given sum; then the rectangle contained by these two lines is the greatest when $P = Q$. Consequently of all the rectangles of a given circumference, the square contains the greatest area.

SECTION CXII.

Prob. *Amongst all the angles which are upon the same base, and have the same circumference, to find that which contains the greatest area.*

Solut. Let the triangle ABC (*fig.* 132) be a maximum; the given base $AC = a$, and with respect to the given circumference $AB + BC = b$.



1. Draw the perpendicular BD: then, since the base AC is determined, this is a maximum. Bisect AC in E, and put $ED = x$, $BD = y$, $AB = z$. Then $AD = \frac{1}{2}a - x$, $CD = \frac{1}{2}a + x$; consequently, since $AB^2 = AD^2 + BD^2$, $BC^2 = CD^2 + BD^2$,

$$z^2 = (\tfrac{1}{2}a - x)^2 + y^2$$
$$(b - z)^2 = (\tfrac{1}{2}a + x)^2 + y^2.$$

2. If the first equation be subtracted from the second, then we obtain

$$b^2 - 2bz = 2ax$$
$$z = \tfrac{1}{2}b - \frac{ax}{b}.$$

3. Substitute this value of z in the first equation in 1; this gives

$$y^2 = \frac{b^2 - a^2}{b^2}(\tfrac{1}{4}b^2 - x^2).$$

4. Hence it follows, that $y = BD$ is the greatest, when $x = 0$, and consequently when the perpendicular BD bisects the line AC. But in this case, $AB = BC$; consequently, of all the triangles of the same circumference and on the same base, the isosceles triangle contains the greatest area.

Cor. This result may likewise be proved geometrically in the following way.

Let ABC (*fig.* 133) be an isosceles triangle, AMC any other triangle on the same base AC, and $AM + MC = AB + BC$: prove that $\triangle ABC > \triangle AMC$.

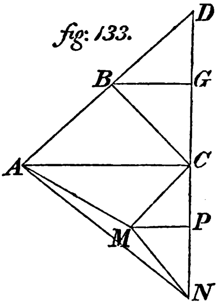


fig: 133.

With B as a centre, and radius $AB = BC$, suppose a circle described, which meets AB produced in D. Draw DC; then the angle ACD, being in a semicircle, $= R$. Produce the perpendicular DC; make $MN = MC$, and draw AN. From the points B, M, draw BG, MP perpendicular to DN.

Since $BC = BD$, and $MN = MC$; then $AB + BC = AD$, and $AM + MC = AM + MN$; consequently $AD = AM + MN$, and $\therefore AD > AN$; and since $CD^2 = AD^2 - AC^2$, $CN^2 = AN^2 - AC^2$, consequently $CD > CN$. Now since $CG = \frac{1}{2} CD$, $CP = \frac{1}{2} CN$: then likewise $CG > CP$. But CG is the altitude of the $\triangle ABC$, and CP the altitude of the $\triangle AMC$; consequently the triangle ABC has a greater altitude, and $\therefore$ also a greater area, than the triangle AMC.

SECTION CXIII.

From the foregoing section, the following general and important rule is deduced:

Of all polygons of equal circumference, and of the same number of sides, that one which is equilateral has the greatest area.

For let $ABCDEF$ (*fig.* 134) be the greatest polygon. If AB be not equal to BC; then upon AC as a base describe an isosceles triangle AMC, whose circumference is equal to that of ABC, in which consequently $AM + MC = AB + BC$. Then by the foregoing section, $\triangle AMC > ABC$, and consequently also the polygon $AMCDEF$, which has the same number of sides, and the same circumference as

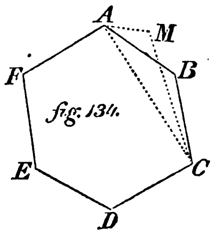


fig: 134.

the polygon $ABCDEF$, is greater than this last. Therefore the polygon $ABCDEF$ is not the greatest of all the polygons having the same circumference and the same number of sides. Consequently $AB = BC$. But, likewise for the same reason, $BC = CD$, $CD = DE$, &c.; consequently the polygon which is a maximum, is equilateral.

SECTION CXIV.

Prob. *Two sides of a triangle are given: find the angle contained by these sides, when the triangle is a maximum.*

Solut. Let (ABC *fig.* 132) be the greatest of all the triangles, which have the same sides AB, BC.

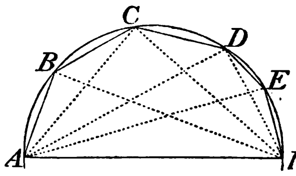
Since $\triangle ABC = \frac{1}{2} AB \,.\, BC \; Sin.\, ABC$, and the lines AB, BC, are given; then that triangle is the greatest, for which $Sin.\, ABC$ is the greatest. But $Sin.\, ABC$ is the greatest, when ABC is a right angle; consequently the greatest triangle is that, the two given sides of which contain a right angle.

SECTION CXV.

Prob. *All the sides but one of a polygon are given: find the conditions under which the polygon is a maximum.*

Solut. Let $ABCDEF$ (*fig.* 135), be the greatest of all the polygons, which are composed of the sides AB, BC, CD, DE, EF, and an undetermined side AF. Draw the diagonals AD, DF: then ADF is a right angle. For if ADF be not a right angle, then, by the foregoing section,

fig:135.



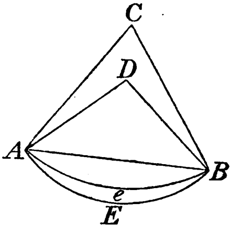
retaining the parts $ABCD$, DEF, the triangle ADF, and consequently also the polygon $ABCDEF$ may be enlarged; which is contrary to the supposition, that this polygon is a

maximum. In like manner it may be proved, that *ABF*, *ACF*, *AEF*, are right angles. Consequently the points *A*, *B*, *C*, *D*, *E*, *F*, must be in a semicircle, whose diameter is *AF*.

Cor. That there can be only one polygon, which, under this condition, can be constructed from the given sides, the following observations will demonstrate.

In the first place it is evident, that when two arcs *AEB*, *AeB* (*fig.* 136), have the same chord *AB*, the chord to which the given radius belongs has a smaller angle at the centre, and consequently, when *C*, *D*, are the centres of these two arcs, $ADB > ACB$.

fig. 136.



Hence it immediately follows, that when once a semicircle (*fig.* 135) is found, in which the given sides *AB*, *BC*, *CD*, *DE*, *EF*, are exactly contained, it is not possible to find another which satisfies these conditions. For if the radius of the other semicircle be greater or less than the radius of the first; then, in the first case, the angles at the centre, which belong to the arcs *AB*, *BC*, &c., would be less than before, and in the second case greater. In both cases these angles cannot together be greater than two right angles, which is the condition.

It matters not in what order the sides *AB*, *BC*, &c. follow each other in the semicircle, since in each series of these sides, the sum of the arcs, which are by them cut off, is always equal to half the circumference; likewise the area of the polygon remains the same, and this last, because the sum of the segments *AB*, *BC*, &c. remains the same.

SECTION CXVI.

From the foregoing section the following rule is deduced:

Prob. *Amongst all the polygons which consist of a certain number of given sides, that one about which a circle can be described is the greatest.*

For let *ABCDEFG* (*fig.* 137) be a polygon described in

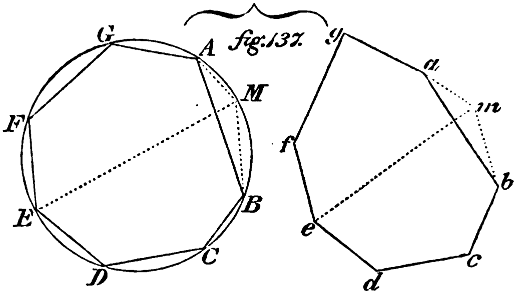


fig. 137.

a circle, and $abcdefg$ another, about which no circle can be described, and whose sides are equal to those of the first, so that $ab = AB$, $bc = BC$. $cd = CD$, and so on. Draw the diameter EM, and the lines AM, BM; upon $ab = AB$, describe the triangle abm, similar and equal to ABM, and draw em. Then by the foregoing section, the polygon $EFGAM$ is greater than the polygon $efgam$, unless indeed this last polygon can be described in a circle, whose diameter is em, in which case, as was shown in the above section, its area is equal to that of the former. For the same reason, and under the same condition, the polygon $EDCBM$, is also greater than $edcbm$. The polygon $AMBCDEFG$ is likewise necessarily greater than $ambcdefg$; for it cannot be equal to it, for otherwise about the whole polygon $ambcdefg$ a circle might be described; which is contrary to the hypothesis. If, now, we take from the above two polygons the equal triangles ABM, abm: then it follows, that the polygon $ABCDEFG$ is greater than the polygon $abcdefg$.

SECTION CXVII.

From §§ CXIII, CXVI, the following rule is deduced:

Of all the polygons of the same circumference and of the same number of sides, the regular polygon is the greatest.

For by § CXIII, the polygon which is a maximum, is equilateral, and by the preceding section, it can be described in a circle: ∴ it is regular.

SECTION CXVIII.

From the foregoing section the following rule is also deduced:

The circle is greater than every rectilineal figure, which has the same circumference.

For it was there proved, that the regular polygon is greater than every other of the same circumference and the same number of sides. If, now, it can be proved, that the circle is always greater than a regular polygon of the same circumference, the truth of the rule follows at once.

Let ∴ r be the radius of a circle, p its circumference, q its area. Assume any regular polygon of the same circumference with the circle; let the perpendicular drawn from its centre to any of its sides $= h$; the area $= q'$. Further, suppose a polygon similar to it described at the circle; let its circumference $= p'$, and the area $= q''$. Then, as may be easily seen,

$$q = \tfrac{1}{2} rp, \quad q' = \tfrac{1}{2} hp, \quad q'' = \tfrac{1}{2} rp'.$$

Because of the similarity of the above two polygons, we also have

$$q'' : q' = r^2 : h^2.$$

If in the proportion for q'', q', their values are substituted, we then obtain

$$\tfrac{1}{2} rp' : \tfrac{1}{2} hp = r^2 : h^2,$$

or $$p' : p = r : h$$

and ∴ $$p' = \frac{pr}{h};$$

consequently $$q'' = \tfrac{1}{2} rp' = \frac{pr^2}{2h}$$

$$q'q'' = \tfrac{1}{4} p^2 r^2 = q^2.$$

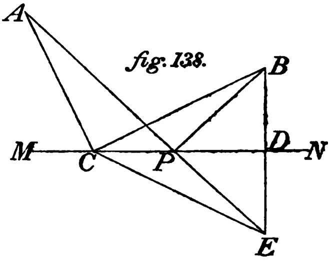
We ∴ have $$q' : q = q : q''.$$

Now since the circle is always less than the polygon described about it, consequently $q'' > q$, and ∴ also $q > q'$. Q. E. D.

SECTION CXIX.

Prob. *In a line given in position, to find a point, such, that when two straight lines are drawn to this point from two given points with this line, their sum is a minimum.*

Solut. Let MN (*fig.* 138) be the line given in position, C a point in it, A, B, the two given points; so that when the lines AB, BC, are drawn, $AC + BC$ is a minimum.



1. From one of the given points, say B, draw BD perpendicular to MN, produce it, and make $DE = BD$; draw CE, AE, and from P, in which AE, MN, intersect each other, draw the line BP.

2. Since $BD = DE$, $CD = CD$, $CDB = CDE = R$; then $\triangle CDB$ is similar and equal to $\triangle CDE$, and $\therefore$ $CB = CE$; consequently $AC + CB = AC + CE$. The sum of the lines AC, CE, and consequently also the sum of the lines AC, CB, will be the least, when the point C falls in P; $\therefore$ $AP + PB$ is a minimum, and consequently P is the point sought.

Cor. Since $BD = DE$, $PD = PD$, $BDP = EDP$; therefore $\triangle BDP$ is similar and equal to $\triangle$ EDP, and $\therefore$ $\angle BPD = \angle DPE$. But $DPE = APM$; consequently $APM = BPD = BPN$. The required point is where the lines AP, BP, make equal angles with the line MN.

SECTION CXX.

Prob. *Two points without a given circle are given; find a point in its circumference such, that when straight lines are drawn from it to the given points, the sum of these two lines is a minimum.*

Solut. Let (*fig.* 139) C be the centre of a circle, M a point on the convex side of its circumference, such, that the angle $AMC = BMC$, and P any other point in the circumference. To M draw the tangent DE; further, draw the lines AP, BP, and from the point Q, in which AP, DE, intersect each other, draw BQ.

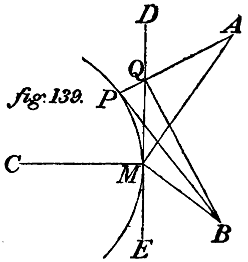


fig. 139.

Since $AMC = BMC$, $CMD = CME = R$: then $AMD = BME$. But $BQE < BME$, $AQD > AMD$; consequently $AQD > BQE$, and ∴ by the foregoing section, $AQ + BQ > AM + BM$. Now $AP + BP > AQ + BQ$ (*Euc.* I. 21); much more ∴ is $AP + BP > AM + BM$. Since this is true, wherever the point P is assumed; consequently M is the point, for which the sum of the lines AM, BM, is a minimum.

Remark. Although the property by which the point M is determined, for which the sum of the lines AM, BM is a minimum, is extremely simple; yet this point cannot be determined in any way by Elementary Geometry. A very elegant solution of this problem is given by Robert Simson, which, on account of conic sections being made use of in it, does not belong to this subject.

SECTION CXXI.

Prob. *In a given triangle to determine a point, which is such, that when lines are drawn from it to the angular points of the triangle, the sum of these lines is a minimum.*

Solut. Let ABC (*fig.* 140) be the given triangle; P the point for which $PA + PB + PC$ is a minimum. From the point C, with the radius CP, suppose a circle described; then $PA + PB$ must necessarily be less than the sum of the lines, which can be drawn from the points A, B, to any other point of the circumference of this circle; because if this be not the case, the sum of the three lines

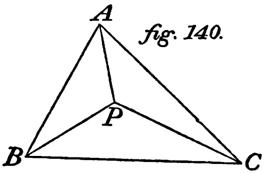


fig. 140.

PA, PB, PC, could not be a minimum. Therefore, by the foregoing section, the angle $APC = BPC$. In like manner it may be shown, that $APB = APC$; $\therefore$ $APB = APC = BPC$, and consequently each angle $= 120^0$.

If one of the angles of the triangle be greater than 120^0; then the problem is impossible.

REMARK. Many more problems relating to Maxima and Minima may be solved by means of Elementary Geometry; there are a great many cases of this kind to be found in a work on this subject, by L. Huilier, entitled: "De Relatione mutua Capcitate et Terminorum Figurarum, geometrice considerata: seu de Maximis et Minimis: Varsarviæ, 1782." Also in his work on Polygonometry, already quoted, p. 174. But however ingenious such elementary solutions for single cases may be, yet they are by no means suited to the establishment of general rules for the treatment of this subject. How far the Differential Calculus, and especially the Calculus of Variations, invented by the celebrated Lagrange (which alone is sufficient to immortalise its inventor) is applicable to this subject, is reserved to the following Collections.

XI LOCI.

SECTION CXXII.

Definition.

When several points have any one property in common, and all are in a straight or crooked line, then this line is called the *Locus* of these points, or of each of them; and a Plane Locus, when the line in which the points lie, is a straight line. The following examples will serve to elucidate what has been said.

Let the base of a triangle, together with its area, be given; find its vertex. The properties which are here required of the triangles sought, apply to an endless number of triangles, whose vertices are all in a straight line, which is parallel to the given base. This line is consequently the Locus of the required vertex.

Let the base and the vertical angle of a triangle be given; required to find its vertex. Here the given properties are evidently not sufficient for the determination of the triangle and its vertex; for there is an endless number of points, which satisfy the conditions of the problem; and all these points are in a circular arc, which has the given base of the required triangle for a chord. Consequently this circular arc is the *Locus* of the required vertex.

SECTION CXXIII.

PROB. *From a given point there is a line drawn, whose extremity touches another straight line given in position: find the Locus of the point which divides the first line in a given proportion.*

SOLUT. From a given point C (*fig.* 141), let any line

CH be drawn to a line *AB* given in position, and let this line be divided according to a certain proportion in *I*: represent the Locus of the point *I*.

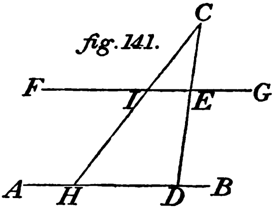


fig. 141.

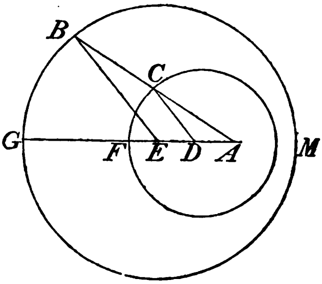
Draw any line *CD*, cut it in the given proportion in *E*, and through this point draw the line *FG* parallel to *AB*: then this line is the *Locus* sought. For if any line *CH* be drawn, which cuts *FG* in *I*, and touches *AB* in *H*; then $CH : CI = CD : CE =$ the given proportion.

SECTION CXXIV.

Prob. *From a given point a straight line is drawn, whose extremity touches the circumference of a given circle: find the Locus of the point which divides this straight line in a given proportion.*

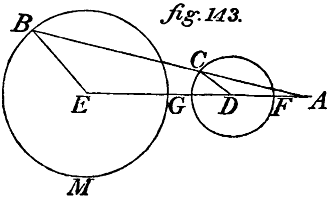
Solut. Let *A* (*figs.* 142, 143) be the given point, *BGM* the given circle, *E* its centre, *AB* any straight line drawn from the point *A*, and this divided in *C*, so that $AB : AC = m : n$; find the Locus of the point *C*.

fig. 142.



Draw the line *AE*, which meets the circumference of the given circle in *G*; determine the point *D*, so that $AE : AD = m : n$; and when this has been found, determine the point *F*, so that $EG : DF = m : n$. Then from *D*, with the radius *DF*, describe a circle: this circle is the required Locus.

fig. 143.



From the point *A* draw any line *AB*, which meets the circumferences of the two circles

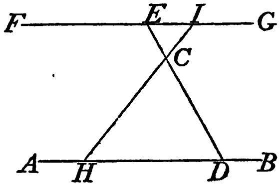
in B, C, and draw CD, BE to their centres. Now since $AE : AD = m : n$, $EG : DF = m : n$; then $AE : AD = EG ; DF = BE : CD$, consequently BE is parallel to CD, and $\therefore$, $AB : AC = AE : AD = m : n$, as required.

SECTION CXXV.

PROB. *From a line given from a certain point, let two parts be cut off by this point in opposite directions, so that these parts may have a given proportion, and that the extremity of one part may meet a line given in position: find the Locus of the extremity of the other part.*

SOLUT. Let C (*fig.* 144) be the given point, AB the straight line given in position, HI any line passing through C which is not given in position, and $CH : CI = m : n$; the extremity H of the line CH touches the line AB: find the Locus of the point I.

fig. 144.



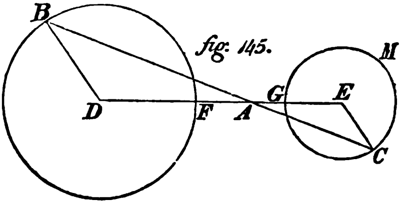
From C to AB draw any line CD, make $CD : CE = m : n$, and through the point E draw the parallel FG: this is the Locus sought. Through C draw any line HI, which meets the lines AB, FG, in H, I; then $CH : CI = CD : CE = m : n$, which was sought.

SECTION CXXVI.

PROB. *In a straight line passing through a given point, two parts are cut off from this point in opposite directions, which have a given proportion; the extremity of one part touches a given circle: find the Locus of the extremity of the other part.*

SOLUT. Let CGM (*fig.* 145) be the given circle, E its

centre, BC any line passing through a given point A, which meets this circle in C; let AB, AC be two parts, whose proportion to one another is constantly $= m : n$; find the Locus of the point B.

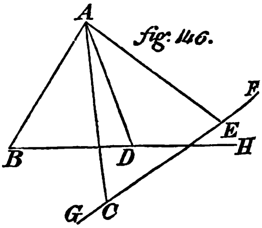


In AE determine a point D, so that $AD : AE = m : n$; then a point F, so that $DF : EG = m : n$; then from D, with a radius DF, describe a circle: this is the required Locus. For if any line BC is drawn through A, and from B, C, in which it meets the two circles, the lines BD, CE be drawn; then, since $AD : AE = m : n$, and $DF : EG = m : n$, likewise $AD : AE = DF : EG = BD : CE$; consequently BD is parallel to CE, and $\therefore$ $AB : AC = AD : AE = m : n$, as required.

SECTION CXXVII.

PROB. *Two straight lines drawn from a given point, and having a given proportion to one another, contain a given angle; the extremity of one line touches a straight line given in position: find the Locus of the extremity of the other line.*

SOLUT. Let A (*fig.* 146) be the fixed point, from which the lines AB, AC are drawn, which contain a given angle BAC, and have a given proportion to one another; the point B meets the line BH; find the Locus of the point C.



From A towards BH, draw any line AD, make the angle DAE equal to the given one, and take AE, so that AD has the given proportion to AE; through the point E draw the line FG, forming the angle $AEG = ADB$: this line is the required Locus. Draw any two

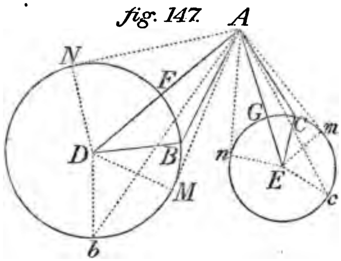
lines AB, AC, forming the given angle $BAC = DAE$; then in the triangles ABD, ACE, the angle $BAD = CAE$ and $ADB = AEC$; consequently triangle ABD is similar to the triangle ACE, and $AB : AC = AD : AE =$ the given proportion, as required.

SECTION CXXVIII.

Prob. *Two straight lines which are drawn from a given point, and have a given proportion, contain a given angle; the extremity of one line touches a given circle: find the Locus of the extremity of the other line.*

Solut. Let A (*fig.* 147) be a given point, D the centre of a circle BMb, and from A let any line AB be drawn to the circumference of this circle; to AB apply an angle BAC of a given magnitude, and upon AC determine the point C, so that $AB : AC$ may be equal to a given proportion $m : n$.

fig. 147.



Draw the line AD, which cuts the circumference of the given circle in F, make the angle DAE equal to the given one, and in AE determine the point E, so that $AD : AE = m : n$, and then the point G, so that likewise $DF : EG = m : n$. If now from the point E, with a radius EG, a circle is described; then this circle will be the required Locus.

For let AB, AC be any two lines drawn to the circumferences of both circles, containing the angle BAC equal to the given one; and draw DB, EC: then $BAC = DAE$, and consequently also, $DAB = EAC$. Further, since $AD : AE = m : n$, and $DF : EG = m : n$; then $AD : AE = DF : EG = DB : EC$. The angles DAB, EAC, are ∴ similar (*Euc. VI.* 7), if it can be shown, that ABD, ACE, are always at the same time obtuse, right, or acute, angles. Assume this to be the case: then $AD : AE = AB : AC$. But $AD : AE = m : n$; consequently also $AB : AC = m : n$, and ∴ the circle, whose centre is E, is the required position.

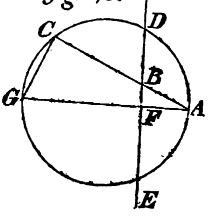
But that the two angles ABD, ACE have the above property, can be proved in the following way. From A draw AM, AN, Am, An tangents to the two circles; and to the points where they touch the circles, the radii DM, DN, Em, En. It may be easily proved, that the triangles DAM, EAm, and consequently also the triangles DAN, EAn, which are equal to them, are similar to one another; for $AD : AE = DM : Em (= m : n)$, and $AMD = AmE = R$. Therefore the angle $DAM = DAN = EAm = EAn$, and consequently also $DAE = MAm = NAn$. If ∴ the point B is situated in M or N, then the point C is in m or n. First suppose that the point B falls in F, and consequently the point C in G; then $ABD = ACE = 2\,R$. Now if the point B remove from F to M, the point C at the same time removes from G to m, the angles ABD, ACE constantly become less, but continue to be obtuse, till B moves into M, and C into m, in which case both are right angles. Beyond M, m, these angles become acute, as AbD, AcE, when b, c are two corresponding points, and remain so, till B, C remove into N, n, where they are right angles, and then again become obtuse. Consequently the condition assumed in the proof, is fulfilled.

SECTION CXXIX.

Prob. *From a line which is drawn from a given point, two parts are cut off, so that their rectangle contained by them has a given magnitude. The extremity of one part meets a straight line given in position: determine the Locus of the extremity of the other part.*

Solut. From the line AC (*fig.* 148), which is drawn from the given point A, two parts AB, BC are cut off such, that the rectangle $AB \times AC$ has a given area $= q$. The point B meets the line DE: determine the Locus of the point C.

fig. 148.



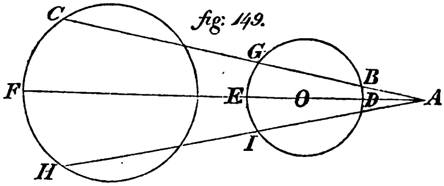
From A draw AF perpendicular to DE; in this line, or this same one produced, determine the point G, so that $AF \times AG = q$; upon AG, as a diameter,

describe a circle: this circle is the position sought. From A draw any line AC, and then CG. Then ACG, as an angle in a semicircle, is a right angle, and $\therefore$ $AFB = ACG$; consequently, because the angle CAG is common to both, $\triangle ACG$ is similar to $\triangle AFB$. We $\therefore$ have $AG : AB = AC : AF$, and consequently $AB \times AC = AG \times AF = q$, as required.

SECTION CXXX.

Prob. *From a line drawn from a given point, two parts are cut off, such, that the rectangle contained by the two parts have a given magnitude. The extremity of one part meets the circumference of a given circle: determine the Locus of the extremity of the other.*

Solut. From the given point A (*fig.* 149), a line AC is



drawn, and from it two parts AB, AC are cut off, so that $AB \times AC$ has an area $= q$. The point B meets a given circle $BDEG$: determine the Locus of the point C.

From A, through the centre O of the given circle, draw the line AO, which cuts its circumference in D, E, and in this line determine a point F such, that $AF \times AD = q$. Then find by § CXXVI, the Locus of a point H, such, that when AH is drawn, and $AH : AI$ is made equal to the given proportion $AF : AE$, the point I meets the given circle. Let the circle CFH be this Locus; then, I assert, this circle is also the Locus of the point C, for which $AC \times AB = q$.

If from A any line AC is drawn, which touches the circumferences of both circles in C and G: then, by the construction, $AC : AG = AF : AE$, and $\therefore$ likewise $AC \times AB : AG \times AB = AF \times AD : AE \times AD$. Now (*Euc.*

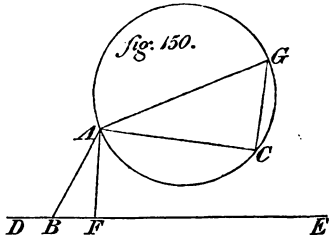
III. 36) $AG \times AB = AE \times AD$; consequently also $AC \times AB = AF \times AD = q$, as required.

Cor. Hence it also follows, that when the point B meets the convex or the concave side of the circle opposite to A, the point C, in the first case, is situated on the concave side, and in the second, on the convex side, of the circle sought.

SECTION CXXXI.

Prob. *From a given point, two straight lines containing a given angle are drawn, and from these two parts are cut off, such, that the rectangle contained by them has a given area. The extremity of one part meets a straight line given in position; find the Locus of the extremity of the other part.*

Solut. From the given point A (*fig.* 150), draw two lines AB, AC, so that the angle BAC has a given magnitude, and the rectangle $AB \times AC$ has a given area $= q$; the point B lies in the straight line DE; find the Locus of the point C.

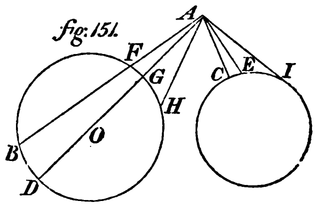


Draw AF perpendicular to DE, make the angle FAG equal to the given one, and $FA \times AG = q$; upon AG as a diameter, describe a circle: then this circle is the Locus sought. From A draw any two lines AB, AC, containing the given angle, the first to DE, the other to the circumference of the circle, and moreover draw the line CG. Then from the construction, $BAC = FAG$, and $\therefore$ likewise $BAF = CAG$. Now since also ACG as an angle in a semicircle, is a right angle; consequently $ACG = AFB$, and $\therefore \triangle ACG$ is similar to $\triangle ABF$. We consequently have $AB : AG = AF : AC$, and $\therefore AB \times AC = AG \times AF = q$, as required.

SECTION CXXXII.

PROB. *From a given point two straight lines containing a given angle, are drawn, such, that the rectangle contained by them has a given area. The extremity of one line is situated in the circumference of a given circle: find the Locus of the extremity of the other line.*

SOLUT. Let BDH (*fig.* 151) be a given circle, O its centre; AB, AC, any two lines drawn from A, which contain a given angle $BAC = \alpha$, and which are such, that the rectangle contained by them has a given area $= q$.



Draw AO, which cuts the given circle in D, G, make $DAE = \alpha$, and in AE determine the point E, so that $AD \times AE = q$. Now let I be a point, which is determined by these means, that when any line AH is drawn to the circle, it makes the angle $HAI = \alpha$, and the proportion $AH : AI$ equal to the given proportion $AG : AE$. The Locus of this point is ∴ a circle, which may be found by § CXXVIII. This circle is also the Locus of the point C.

According to the construction, for every two lines drawn from A to the two circles, which contain an angle $= \alpha$, viz. AF, AC, $AF : AC = AG : AE$; consequently also $AF : AB : AC : AB = AG \times AD : AE \times AD$. Now $AF \times AB = AG \times AD$; consequently likewise $AC \times AB = AE \times AD = q$, as required.

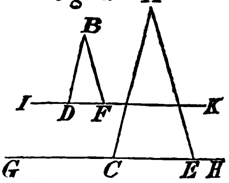
SECTION CXXXIII.

PROB. *From two given points two parallel lines are drawn, which have a given proportion. The extremity*

of one meets a straight line given in position: find the Locus of the extremity of the other.

Solut. Let A, B (*fig.* 152) be two given points, and AC, BD, two parallel lines; the point C is in the given line GH: find the Locus of the point D, when for each two of these lines, the proportion $AC : BD$ is equal to the given proportion $m : n$.

fig. 152.



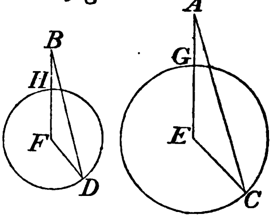
From A to GH draw any line AE, and from B draw BF parallel to AE; make $AE : BF = m : n$, and through the point F thus determined, draw IK parallel to GH: then this line is the required Locus. From A, B, to the lines GH, IK, draw any two parallels AC, BD: then, since AE is parallel to BF, AC parallel to BD, CE parallel to DF, the three angles of the triangle ACE are equal to the three angles of the triangle BDF, and $\therefore$ $AC : BD = AE : BF = m : n$, as required.

SECTION CXXXIV.

Prob. *From two given points, two parallel lines are drawn of a given proportion; the extremity of one is in the circumference of a given circle: find the Locus of the extremity of the other.*

Solut. Let A, B (*fig.* 153) be two given points; AC, BD, two parallel lines, which have the given proportion $m : n$; the point C is in the circumference of the circle, whose centre is E: find the Locus of the point D.

fig. 153.



Draw the line AE and BF parallel to it, and determine the point F, so that $AE : BF = m : n$; then make $EG : FH = m : n$, and by these means determine the point H. From F, with the radius FH, describe a circle: then this circle is the required

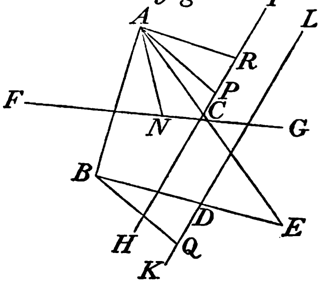
Locus. Draw any two parallel lines AC, BD, to the circumference of the circles, and the radii EC, FD; then the angle $EAC = FBD$; further, $AE : BF = m : n$, and $EC : FD = EG : FH = m : n$; $\therefore AE : BF = EC : FD$. Moreover, as is easily seen, the angles AEC, BFD, are at the same time acute, rectangular, or obtuse. Therefore triangle AEC is similar to triangle BFD, and consequently $AC : BD = AE : BF = m : n$, as required.

SECTION CXXXV.

PROB. *From two given points two straight lines are drawn, which contain a given angle; in each of these a part is cut off from the given point, such, that these parts have a given proportion. The extremity of one part meets a straight line given in position: find the Locus of the extremity of the other.*

SOLUT. Let A, B (*fig.* 154), be two given points, and AE, BE, any two lines, which contain a given angle $AEB = \alpha$. From these lines two parts AC, BD, are cut off, such, that $AC : BD = m : n$. The point C is in the line FG; determine the Locus of the point D.

fig. 154.



From A to FG draw any line AN, make $NAP = \alpha$, and $AN : AP = m : n$, and then determine, by § CXXVII, the Locus of the point P. Let the straight line HI be this Locus. Make the line BQ equal and parallel to AP; further, through the point Q draw the line KL parallel to HI: then this line is the Locus sought. Thus, if any two lines AE, BE are drawn, which contain the given angle α, the parts AC, BD cut off by the lines FG, KL, have the given proportion $m : n$.

Make the angle $CAR = \alpha$, and $AC : AR = m : n$; then, by the construction, the point R is in the line HI. Now

since AP is parallel to BQ, PR parallel to QD, and because the angles AEB, CAR, are equal, then likewise BD is parallel to AR; further, $AP = BQ$, then $\triangle APR$ is similar and equal to $\triangle BQD$, and $AR = BD$. But $AC : AR = m : n$; consequently also $AC : BD = m : n$, as required.

REMARK. To those of my readers who wish to know more of the subject of Plane Loci, I beg to recommend a work already quoted once, viz. the Translation of the Plane Loci of Apollonius, by Camerer. To the ancients these Loci were the chief means of solving problems; but the great improvement in Analysis has rendered them in a great measure unnecessary now-a-days. They may, however, be considered as useful in preparing the student for the higher geometry, and this is the reason why they have not been altogether passed over in this Collection.

XII. MISCELLANEOUS PROBLEMS.

SECTION CXXXVI.

AUXILIARY RULE.

When from the three angular points of a triangle, perpendiculars are drawn to the opposite sides; these perpendiculars intersect each other in one and the same point.

Proof. Let ABC (*figs.* 155, 156) be any triangle; BD,

fig: 155.

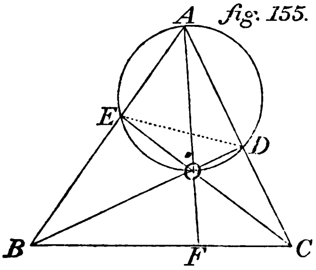
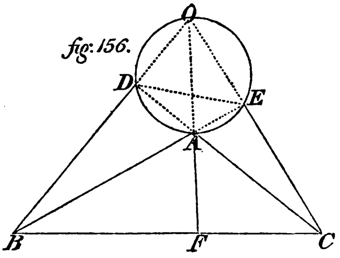


fig: 156.



CE, two perpendiculars to AC, AB, or these produced, and O the point in which BD, CE, intersect each other: it remains to be proved, that when AO is drawn, and produced, it cuts the side BC at right angles in F. Draw DE.

The triangles BOE, COD, are similar, for $BOE = COD$, and $BEO = CDO = R$. We $\therefore$ have $BO : CO = OE : OD$, and since likewise the angle $BOC = EOD$: then $\triangle BOC$ is similar to $\triangle EOD$, and angle $CBD = DEO$. Further, since AEO, ADO, are right angles: then $ADO + AEO = 2R$; consequently a circle may be described

about the quadrilateral $ADOE$, and we have $\angle\ DEO = \angle\ DAO$ (*Euc.* III. 27) and $\therefore CBD = DAO$. The triangles CAF, CBD, consequently have the equal angles CAF, CBD, and the common one ACB; $\therefore$ likewise $AFC = BDC = R$.

Q. E. D.

SECTION CXXXVII.

Prob. *In a triangle, two perpendiculars, drawn from the angular points to the opposite sides, are given: find its sides, angles, and area.*

Solut. In the triangle ABC (*figs.* 155, 156), the perpendiculars given are $BD = a$, $CE = b$, $AF = c$, and the sides sought, are $AB = x$, $AC = y$, $BC = z$. Because the triangles ABD, ACE, are similar, we have $BD : AB = CE : AC$, or $a : x = b : y$, and $\therefore$

$$y = \frac{bx}{a}.$$

In like manner, from the similar triangles CBE, ABF, we find

$$z = \frac{bx}{c}.$$

2. We $\therefore$ have

$$\triangle\ ABC = \tfrac{1}{4}\sqrt{(x+y+z)(x+y-z)(x+z-y)(y+z-x)}$$
$$= \frac{x^2}{4a^2c^2}\sqrt{(ac+bc+ab)(ac+bc-ab)(ac+ab-bc)(bc+ab-ac)},$$

or when, for the sake of brevity, we put

$$(ac+bc+ab)(ac+bc-ab)(ac+ab-bc)(bc+ab-ac) = K$$

we get,

$$\triangle\ ABC = \frac{x^2\sqrt{K}}{4a^2c^2}.$$

3. But likewise $\triangle\ ABC = \tfrac{1}{2} AB \times CE = \tfrac{1}{2} bx$; we

consequently have the equation

$$\frac{x^2 \sqrt{K}}{4a^2c^2} = \tfrac{1}{2}bx,$$

whence we obtain

$$x = \frac{2a^2bc^2}{\sqrt{K}}.$$

Hence we obtain the two remaining sides of the triangle: thus,

$$y = \frac{bx}{a} = \frac{2ab^2c^2}{\sqrt{K}}$$

$$z = \frac{bx}{c} = \frac{2a^2b^2c}{\sqrt{K}}.$$

4. For the angles of the triangle, we find

$$Sin.\ BAC = \frac{a}{x} = \frac{\sqrt{K}}{2abc^2}$$

$$Sin.\ ABC = \frac{c}{x} = \frac{\sqrt{K}}{2a^2bc}$$

$$Sin.\ ACB = \frac{c}{y} = \frac{\sqrt{K}}{2ab^2c}.$$

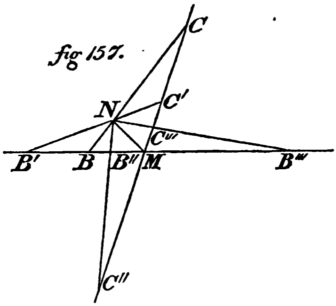
5. Further,

$$\triangle\ ABC = \tfrac{1}{2}bx = \frac{a^2b^2c^2}{\sqrt{K}}.$$

SECTION CXXXVIII.

Prob. *Two lines intersect each other at a given angle; this angle is bisected by a line, in which a point is given: it is required through this point to draw a line, such, that the part contained between the two first lines intersecting each other, may have a given magnitude.*

Solut. The lines $B'B'''$, CC'' (*fig.* 157), intersect each



other in M, forming the angle $BMC = \alpha$; this angle is bisected by the line MN, and $MN = a$; draw a line through N, so that the line contained between the two first lines, viz. $BC = b$.

1. Since $BMO = \alpha$: therefore $BMN = CMN = \frac{1}{2}\alpha$. If the angle BNM is also known, then the problem is solved. Let ∴ $BNM = \phi$; $MCN = \phi - \frac{1}{2}\alpha$, $MBN = 180^0 - (\phi + \frac{1}{2}\alpha)$. We ∴ have in the triangle BMN,

$$BN = \frac{MN \; Sin. \; BMN}{Sin. \; MBN} = \frac{a \; Sin. \; \frac{1}{2}\alpha}{Sin. \; (\phi + \frac{1}{2}\alpha)},$$

and in the triangle CMN,

$$CN = \frac{MN \; Sin. \; CMN}{Sin. \; MCN} = \frac{a \; Sin. \; \frac{1}{2}\alpha}{Sin. \; (\phi - \frac{1}{2}\alpha)},$$

Now, since $BC = BN + CN$; we ∴ have the equation,

$$b = \frac{a \; Sin. \; \frac{1}{2}\alpha}{Sin. \; (\phi + \frac{1}{2}\alpha)} = \frac{a \; Sin. \; \frac{1}{2}\alpha}{Sin. \; (\phi - \frac{1}{2}\alpha)},$$

or

$$b \; Sin. \; (\phi + \tfrac{1}{2}\alpha) \; Sin. \; (\phi - \tfrac{1}{2}\alpha) = a \; Sin. \; \tfrac{1}{2}\alpha \left[Sin. \; (\phi + \tfrac{1}{2}\alpha) + Sin. \; (\phi - \tfrac{1}{2}\alpha)\right]$$

2. But we have $Sin. \; (\phi + \frac{1}{2}\alpha) \; Sin. \; (\phi - \frac{1}{2}\alpha) = Sin.^2 \phi - Sin.^2 \frac{1}{2}\alpha$, and $Sin. \; (\phi + \frac{1}{2}\alpha) + Sin. \; (\phi - \frac{1}{2}\alpha) = 2 \; Sin. \; \phi \; Cos. \; \frac{1}{2}\alpha$; consequently the foregoing equation is transformed into the following one:

$$b \; Sin.^2 \phi - b \; Sin.^2 \tfrac{1}{2}\alpha = 2 a \; Sin. \; \tfrac{1}{2}\alpha \; Cos. \; \tfrac{1}{2}\alpha \; Sin. \; \phi,$$

or, since $2 \; Sin. \; \frac{1}{2}\alpha \; Cos. \; \frac{1}{2}\alpha = Sin. \; \alpha$, into

$$Sin.^2 \phi - \frac{a \; Sin. \; \alpha}{b} \; Sin. \; \phi = Sin.^2 \tfrac{1}{2}\alpha;$$

and hence we obtain

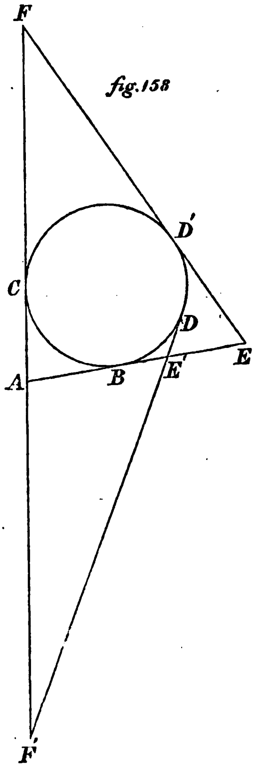
$$Sin.\ \phi = \frac{a\ Sin.\ \alpha \pm \sqrt{(a^2\ Sin.^2\ \alpha + 4b^2\ Sin.^2\ \tfrac{1}{2}\ \alpha)}}{2\ b}.$$

REMARK. The problem in § LXXXIII is only a particular case of this more general one. What was there said of the four solutions of the problem there given, also applies to this one; for each sine has two angles. The figure shows the four different positions which the line *BC* can have.

SECTION CXXXIX.

PROB. *Two tangents are drawn to a given circle: find a third tangent such, that the part contained between the two first has a given magnitude.*

SOLUT. Let *BDC* (*fig.* 158) be a given circle; *AB*, *AC*, two of its tangents, so that $AB = AC = a$, and the angle $BAC = \alpha$; draw a third tangent *EF*, which cuts the two first in *E*, *F*, so that $EF = b$.



1. If *DE* is known, then the problem is solved; for it would only be necessary in this case to make $BE = DE$, and from the point *E* so determined to draw a tangent to the circle. Put ∴ $DE = x$; this gives $DF = b - x$, $AE = AB + BE = a + x$, $AF = AC + CF = AC + DF = a + b - x$.

2. In the triangle *AEF*,

$$EF^2 = AE^2 + AF^2 - 2\ AE\ .\ AF\ Cos.\ EAF$$

and ∴

$$b^2 = (a+x)^2 + (a+b-x)^2 - 2(a+x)(a+b-x) \times Cos.\ \alpha$$

or

$$x^2 - bx + (a^2 + ab)\frac{1 - Cos.\ \alpha}{1 + Cos.\ \alpha} = 0.$$

From this equation we further obtain

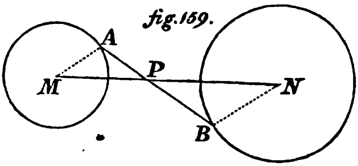
$$x = \tfrac{1}{2} b \pm \sqrt{\left[\tfrac{1}{4} b^2 - (a^2 + ab)\frac{1 - Cos.\ \alpha}{1 + Cos.\ \alpha}\right]}$$
$$- \tfrac{1}{2} b \pm \sqrt{[\tfrac{1}{4} b^2 - 2 (a^2 + ab)\ Tan.^2\ \tfrac{1}{2}\alpha]},$$

The two expressions just found for x, are always positive, as the figure indeed shows. Thus the line EF can also have the position $E'F'$, and then DE' has the second value.

SECTION CXL.

PROB. *Two circles, and a point in the line which joins their centres, are given : through this point draw a line which meets the circumferences of both circles, such, that the parts included between these circumferences and the given point have a given proportion.*

SOLUT. Let M, N (*fig.* 159) be the centres of the two given circles, and P a point in the line MN; further, let $MP = a$, $NP = b$, the radius $MA = r$, the radius $NB = R$: through P draw a line AB to the circumferences of the two circles, so that $AP : BP = m : n$.



1. If the line AP is determined, the problem is solved. Put ∴ $AP = x$: this gives $BP = \frac{nx}{m}$. Therefore in the triangle AMP,

$$Cos.\ APM = \frac{MP^2 + AP^2 - MA^2}{2\ MP\ .\ AP} = \frac{a^2 + x^2 - r^2}{2\ ax}$$

and in the triangle BNP,

$$Cos.\ BPN = \frac{NP^2 + BP^2 - NB^2}{2\ NP\ .\ BP} = \frac{b^2 + \frac{n^2 x^2}{m^2} - R^2}{\frac{2\ bnx}{m}}$$

$$= \frac{m^2 b^2 + n^2 x^2 - m^2 R^2}{2\ mnbx}.$$

2. Now since *Cos.* APM = *Cos.* BPN: we obtain the following equation:

$$\frac{a^2 + x^2 - r^2}{2ax} = \frac{m^2b^2 + n^2x^2 - m^2R^2}{2mnbx},$$

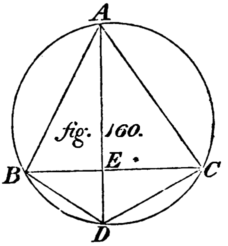
and hence

$$x = \sqrt{\frac{am^2(b^2 - R^2) - bmn(a^2 - r^2)}{bmn - an^2}}$$

SECTION CXLI.

PROB. *From two given points in the circumference of a given circle, to draw two chords, which contain a given angle, and have a given proportion.*

SOLUT. Let $ABDC$ (*fig.* 160) be a given circle, whose radius $= r$; let A, B, be two points in the circumference of the circle, from which two chords AB, BC, are drawn, so that $AEC = \alpha$, and $AD : BC = m : n$. Further, let $AB = a$.



1. If we know the angles BAD, DAC: we then can draw the chords AD, DC. Put $\therefore$ $BAD = \phi$, $DAC = \psi$. Hence we find $CBD = CAD = \psi$, $ABC = AEC - BAD = \alpha - \phi$, $ACB = 180^0 - (AEC + CAD) = 180^0 - (\alpha + \psi)$, $BAC = BAD + DAC = \phi + \psi$, $ABD = ABC + CBD = \alpha - \phi + \psi$.

2. By the first principles of Trigonometry, $AB = 2r \times$ *Sin.* $ACB = a$, $AD = 2r$ *Sin.* ABD, $BC = 2r$ *Sin.* BAC, and $\therefore$ $AD : BC =$ *Sin.* ABD : *Sin.* $BAC = m : n$. Now, if for ACB, ABD, BAC, we put their values from 1, we then obtain the two following equations:

$$2r \, Sin. \, (\alpha + \psi) = a$$
$$m \, Sin. \, (\phi + \psi) = n \, Sin. \, (\alpha - \phi + \psi).$$

3. In order to determine from hence the angles ϕ, ψ,

expand $Sin. (\alpha+\psi)$ in the first equation; this gives

$$2\,r\,(Sin.\,\alpha\; Cos.\,\psi + Cos.\,\alpha\; Sin.\,\psi) = a$$

or

$$2\,r\; Cos.\,\alpha\; Sin.\,\psi = a - 2\,r\; Sin.\,\alpha\; Cos.\,\psi;$$

and when the square root is extracted from both sides of the equation, and $1 - Cos.^2\,\phi$ is substituted for $Sin.^2\,\phi$,

$$4\,r^2\; Cos.^2\,\alpha - 4\,r^2\; Cos.^2\,\alpha\; Cos.^2\,\psi = a^2 - 4\,a\,r\; Sin.\,\alpha\; Cos.\,\psi \\ + 4\,r^2\; Sin.^2\,\alpha\; Cos.^2\,\psi$$

or

$$4\,r^2\; Cos.^2\,\psi - 4\,a\,r\; Sin.\,\alpha\; Cos.\,\psi = 4\,r^2\; Cos.^2\,\alpha - a^2,$$

whence we obtain

$$Cos.\,\psi = \frac{a\; Sin.\,\alpha}{2\,r} \pm \frac{Cos.\,\alpha}{2r}\sqrt{(4\,r^2 - a^2)};$$

and this equation serves to determine the angle ψ.

4. Multiply this equation by $2\,r\; Sin.\,\alpha$; this gives

$$2\,r\; Sin.\,\alpha\; Cos.\,\psi = a\; Sin.^2\,\alpha \pm Sin.\,\alpha\; Cos.\,\alpha\sqrt{(4\,r^2 - a^2)}.$$

But from 3,

$$2\,r\; Sin\,\alpha\; Cos.\,\psi + 2\,r\; Cos.\,\alpha\; Sin.\,\psi = a;$$

if ∴ we subtract the first from the second, we then obtain

$$2\,r\; Cos.\,\alpha\; Sin.\,\psi = a\; Cos.^2\,\alpha \mp Sin.\,\alpha\; Cos.\,\alpha\sqrt{(4\,r^2 - a^2)}$$

and

$$Sin.\,\psi = \frac{a\; Cos.\,\alpha}{2\,r} \mp \frac{Sin.\,\alpha}{2\,r}\sqrt{(4\,r^2 - a^2)}.$$

5. In order now to determine the angle ϕ, also expand the second equation in 2; this gives

$$m\,[Sin.\,\phi\; Cos.\,\psi + Cos.\,\phi\; Sin.\,\psi] = n\,[Sin.\,(\alpha + \psi)\; Cos.\,\phi \\ - Cos.\,(\alpha + \psi)\; Sin.\,\phi],$$

and hence we obtain

$$\frac{Sin.\,\phi}{Cos.\,\phi} = Tan.\,\phi = \frac{n\; Sin.\,(\alpha + \psi) - m\; Sin.\,\psi}{m\; Cos.\,\psi + n\; Cos.\,(\alpha + \psi)}.$$

But from the first equation in 2, we obtain $Sin. (\alpha + \psi) = \frac{a}{2r}$; $\therefore Cos. (\alpha + \psi) = \frac{\sqrt{(4r^2 - a^2)}}{2r}$, and when we substitute these values in the expression for $Tan. \phi$,

$$Tan. \phi = \frac{na - 2mr\, Sin. \psi}{2mr\, Cos. \psi + n\sqrt{(4r^2 - a^2)}}.$$

In this expression it is only necessary to substitute for $Sin. \psi$ and $Cos. \psi$ the values found in 3, 4, and we also get the angle ϕ.

REMARK. When general expressions for the angles ϕ, ψ, are not treated of, but merely the actual calculation of a single case; then the following method will be the easiest. Thus from the first equation in 2, we obtain $Sin. (\alpha + \psi) = \frac{a}{2r}$; hence, when numbers only are considered, the angle $\alpha + \psi$, and consequently also ψ may be determined. After ψ has been determined in this way, the equation in 5, viz.

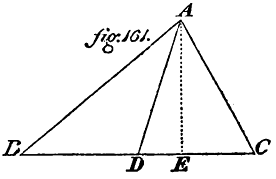
$$Tan. \phi = \frac{n\, Sin. (\alpha + \psi) - m\, Sin. \psi}{m\, Cos. \psi + n\, Cos. (\alpha + \psi)}$$

immediately gives the angle ϕ.

SECTION CXLII.

PROB. *The base of a triangle, the difference of the two angles, at this base, and also the line drawn from the vertical angle of the triangle to the centre of the base, are given: find the triangle.*

SOLUT. Let ABC (*fig.* 161) be the required triangle, the base $BC = a$, D the middle of BC, and $AD = b$; further, let $ACB - ABC = \alpha$.



1. If the angle ADC is known; then in each of the triangles CAD, DAB, two sides, and the angle contained by them, are given, consequently these triangles

themselves, and $\therefore$ also the whole triangle BAC. Put $\therefore$ $ADC = \phi$, and draw the perpendicular AE. Then $AE = b\ Sin.\ \phi$, $DE = b\ Cos.\ \phi$, $BE = \frac{1}{2}a + b\ Cos.\ \phi$, $CE = \frac{1}{2}a - b\ Cos.\ \phi$; consequently

$$Tan.\ ABC = \frac{AE}{BE} = \frac{b\ Sin.\ \phi}{\frac{1}{2}a + b\ Cos.\ \phi}$$

$$Tan.\ ACB = \frac{AE}{CE} = \frac{b\ Sin.\ \phi}{\frac{1}{2}a - b\ Cos.\ \phi}.$$

2. By

$$Tan.\ (ACB - ABC) = Tan.\ \alpha = \frac{Tan. ACB - Tan. ABC}{1 + Tan. ACB\, Tan. ABC}.$$

If in this for $Tan.\ ACB$, $Tan.\ ABC$, we substitute their values found in 1, we obtain the equation

$$Tan.\ \alpha = \frac{2\ b^2\ Sin.\ \phi\ Cos.\ \phi}{\frac{1}{4}a^2 - b^2(Cos.^2\ \phi - Sin.^2\ \phi)},$$

or, since $2\ Sin.\ \phi\ Cos.\ \phi = Sin.\ 2\ \phi$, $Cos.^2\ \phi - Sin.^2\ \phi = Cos.\ 2\ \phi$.

$$Tan.\ \alpha = \frac{b^2\ Sin.\ 2\ \phi}{\frac{1}{4}a^2 - b^2\ Cos.\ 2\ \phi}.$$

3. In the equation last found, substitute $\frac{Sin.\ \alpha}{Cos.\ \alpha}$ for $Tan.\ \alpha$; by these means it is transformed into the following one:

$$b^2\ (Sin.\ 2\ \phi\ Cos.\ \alpha + Cos.\ 2\ \phi\ Sin.\ \alpha) = \frac{1}{4}a^2\ Sin.\ \alpha,$$

or

$$b^2\ Sin.\ (2\ \phi + Cos.\ \alpha) = \frac{1}{4}\ a^2\ Sin.\ \alpha;$$

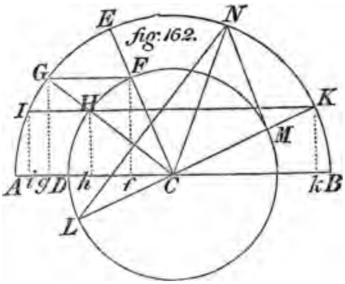
and the equation gives

$$Sin.\ (2\ \phi + \alpha) = \frac{a^2\ Sin.\ \alpha}{4\ b^2}.$$

Hence we may now determine $2\ \phi + \alpha$, and consequently also ϕ.

Cor. From this analytical solution we derive the following tolerably easy construction.

From any point C (*fig.* 162), with a radius $AC = b$, describe the semicircle ANB, and with the radius $CD = \frac{1}{2}a$, the circle LFM; make $ACE = \alpha$, and from the point F, in which the line CE cuts the circle, draw FG parallel to AB, which meets the semicircle in G. Draw CG, and through the point H, in which the circle is cut by this line, draw IK parallel to AB till it meets the semicircle; then draw KC, cutting the circle in L, M; bisect the angle ECK by CN, and from the point N, where it meets the semicircle, draw the lines NL, NM: then LNM is the triangle sought.



Draw the perpendiculars Ff, Gg, Hh, Ii, Kk; then $Gg = Ff = \frac{1}{2}a\,Sin.\,\alpha$. Further, $CG : CH = Gg : Hh$, or $b : \frac{1}{2}a = \frac{1}{2}a\,Sin.\,\alpha : Hh$, and $\therefore\ Hh = \dfrac{a^2\,Sin.\,\alpha}{4\,b} = Kk$, and $Sin.\ BCK = \dfrac{Kk}{CK} = \dfrac{a^2\,Sin.\,\alpha}{4\,b^2} = Sin.\ ACK$. But likewise by the analytical solution, $Sin.\,(2\,\phi + \alpha) = \dfrac{a^2\,Sin.\,\alpha}{4\,b^2}$; $\therefore\ 2\,\phi + \alpha = ACK$, and $\phi = \frac{1}{2}\,ECK = NCM$. The rest is self-evident.

Besides the angle ACK, likewise the angle $ACI = 2\,\phi + \alpha$, $\therefore$ the angle $ICE = -\,2\,\phi$. But since there was no negative angle required here for ϕ, consequently in this case $KCN = \phi$ only.

SECTION CXLIII.

Prob. *To describe a square in a given quadrilateral.*

Solut. Let $ABCD$ (*fig.* 163) be the given quadrilateral, $MNPQ$ the square sought.

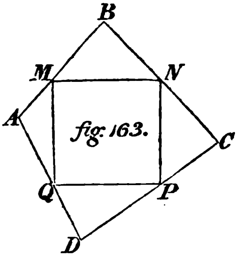


fig. 163.

1. Since the quadrilateral is given: then also are its four angles, A, B, C, D; and the sides AB, BC are known. Let $\therefore AB=a$, $BC=b$. If now we knew the sides of the square, and only one of the angles which its sides make with the sides of the quadrilateral, then all the rest would be determined, and we could describe the square. Put $\therefore MN=x$, $BMN = \phi$.

2. From ϕ and the given angles, all the other angles of the figure may be found. For since $QMN = R$: then $AMQ = 90^0 - \phi$, and $\therefore AQM = 180^0 - A - AMQ = 90^0 - (A - \phi)$. Further, $BNM = 180^0 - (B + \phi)$, and $\therefore$, because MNP is a right angle, $CNP = B + \phi - 90^0$, and $CPN = 180^0 - C - CNP = 270^0 - (B + C + \phi)$. The other angles are not required in this case.

3. The triangle BMN gives

$$BM = \frac{MN\ Sin.\ BNM}{Sin.\ MBN} = \frac{x\ Sin.\ (B + \phi)}{Sin.\ B},$$

$$BN = \frac{MN\ Sin.\ BMN}{Sin.\ MBN} = \frac{x\ Sin.\ \phi}{Sin.\ B}.$$

The triangle MAQ gives

$$AM = \frac{MQ\ Sin\ AQM}{Sin.\ MAQ} = \frac{x\ Cos.\ (A - \phi)}{Sin.\ A},$$

and the triangle NCP,

$$CN = \frac{NP\ Sin.\ CPN}{Sin.\ NCP} = -\frac{x\ Cos.\ (B + C + \phi)}{Sin.\ C}.$$

4. Now since $AM + BM = AB = a$, $BN + CN = BC = b$; we then have the two equations

$$\frac{x\ Cos.\ (A - \phi)}{Sin.\ A} + \frac{x\ Sin.\ (B + \phi)}{Sin.\ B} = a,$$

$$\frac{x\ Sin.\ \phi}{Sin.\ B} - \frac{x\ Cos.\ (B + C + \phi)}{Sin.\ C} = b,$$

and hence, by eliminating x, we obtain

$$a\,Sin.\,A\,Sin.\,C\,Sin.\,\phi - a\,Sin.\,A\,Sin.\,B\,Cos.\,(B+C+\phi)$$
$$= b\,Sin.\,B\,Sin.\,C\,Cos.(A-\phi) + b\,Sin.\,A\,Sin.\,C\,Sin.\,(B+\phi).$$

5. If we solve $Cos.\,(B+C+\phi)$, $Cos.\,(A-\phi)$, $Sin.\,(B+\phi)$, and take away ϕ from the remaining magnitudes; this gives

$$Sin.\,\phi \begin{bmatrix} a\,Sin.\,A\,Sin.\,C + a\,Sin.\,A\,Sin.\,B\,Sin.\,(B+C) \\ -b\,Sin.\,A\,Sin.\,B\,Sin.\,C - b\,Sin.\,A\,Sin.\,C\,Cos.\,B \end{bmatrix}$$
$$= Cos.\,\phi \begin{bmatrix} b\,Sin.\,A\,Sin.\,B\,Sin.\,C + b\,Sin.\,B\,Sin.\,C\,Cos.\,A \\ + a\,Sin.\,A\,Sin.\,B\,Cos.\,(B+C) \end{bmatrix}$$

and hence we obtain

$$Tan.\,\phi = \frac{\begin{bmatrix} b\,Sin.\,A\,Sin.\,B\,Sin.\,C + b\,Sin.\,B\,Sin.\,C\,Cos.\,A \\ + a\,Sin.\,A\,Sin.\,B\,Cos.\,(B+C) \end{bmatrix}}{\begin{bmatrix} a\,Sin.\,A\,Sin.\,C + a\,Sin.\,A\,Sin.\,B\,Sin.\,(B+C) \\ - b\,Sin.\,A\,Sin.\,B\,Sin.\,C - b\,Sin.\,A\,Sin.\,C\,Cos.\,B \end{bmatrix}}$$

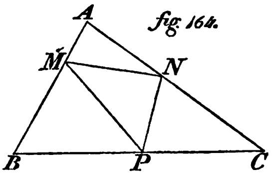
or, if the numerator and denominator be divided by $Sin.\,A$,

$$Tan.\,\phi = \frac{Sin.\,B\,[b\,Sin.\,C + b\,Sin.\,C\,Cot.\,A + a\,Cos.\,(B+C)]}{a\,Sin.\,C + a\,Sin.\,B\,Sin.\,(B+C) - b\,Sin.\,B\,Sin.\,C - b\,Sin.\,C\,Cos.\,B}.$$

SECTION CXLIV.

PROB. *In a given triangle to describe another given triangle.*

SOLUT. Let ABC (*fig.* 164) be the given triangle, in which the triangle MNP, which is also given, is so inscribed, that its angular points touch the sides of the former.



1. Since the two triangles are given; we then likewise know their angles and sides. Let ∴. $BAC = A$, $ABC = B$, $ACB = C$, $NMP = m$, $MNP = n$, $AB = a$, $MN = f$,

$MP = g$. If we know the angle AMN; then the problem is solved. Let $\therefore$ $ANM = \phi$. From this and the given angles we may now determine all the rest. Thus we have $ANM = 180^0 - (A + \phi)$, $BMP = 180^0 - (NMP + AMN) = 180^0 - (m + \phi)$, $BPM = 180^0 - BMP - MBP = m - B + \phi$.

2. Hence we further get,

$$AM = \frac{MN \text{ Sin. } ANM}{\text{Sin. } MAN} = \frac{f \text{ Sin. } (A + \phi)}{\text{Sin. } A},$$

$$BM = \frac{MP \text{ Sin. } BPM}{\text{Sin. } MBP} = \frac{g \text{ Sin. } (m - B + \phi)}{\text{Sin. } B}.$$

3. Now since $AB = AM + BM$: we consequently have the equation

$$\frac{f \text{ Sin. } (A + \phi)}{\text{Sin. } A} + \frac{g \text{ Sin. } (m - B + \phi)}{\text{Sin. } B} = a,$$

or, when we expand Sin. $(A + \phi)$, Sin. $(m - B + \phi)$, and arrange the expressions properly,

$$[f \text{ Sin. } A \text{ Sin. } B + g \text{ Sin. } A \text{ Sin. } (m - B)] \text{ Cos. } \phi +$$
$$[f \text{ Sin. } B \text{ Cos. } A + g \text{ Sin. } A \text{ Cos. } (m - B)] \text{ Sin. } \phi$$
$$= a \text{ Sin. } A \text{ Sin. } B,$$

and when we divide this equation by Sin. A,

$$[f \text{ Sin. } B + g \text{ Sin. } (m - B)] \text{ Cos. } \phi +$$
$$[f \text{ Sin. } B \text{ Cot. } A + g \text{ Cos. } (m - B)] \text{ Sin. } \phi$$
$$= a \text{ Sin. } B$$

4. Divide this equation by f Sin. B Cot. A + g Cos. $(m - B)$, and put

$$\frac{f \text{ Sin. } B + g \text{ Sin. } (m - B)}{f \text{ Sin. } B \text{ Cot. } A + g \text{ Cos. } (m - B)} = \text{Tan. } \mu;$$

$\therefore$ find an angle μ such, that its tangent is equal to the expression on the left side of this equation (a method which has already been frequently used for shortening the calculation): we then obtain

$$\text{Tan. } \mu \text{ Cos. } \phi + \text{Sin. } \phi = \frac{a \text{ Sin. } B}{f \text{ Sin.} B \text{ Cot.} A + g \text{ Cos. } (m - B)}$$

or,

$$Sin.\,\mu\, Cos.\,\phi + Cos.\,\mu\, Sin.\,\phi = \frac{a\ Sin.\ B\ Cos.\ \mu}{f Sin.\, B\, Cot. A + g Cos.\,(m - B)}$$

or lastly

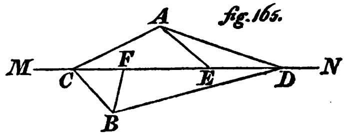
$$Sin.\ (\phi + \mu) = \frac{a\ Sin.\ B\ Cos.\ \mu}{f\ Sin.\ B\ Cot.\ A\ +\ g\ Cos.\ (m\ -\ B)}.$$

From this equation we may now very easily determine $\phi + \mu$, and consequently likewise ϕ.

SECTION CXLV.

Prob. *Two points and a straight line are given: find two points in this line, such, that when lines are drawn from these to the former, the angles which these lines contain at the given points, have a given magnitude.*

Solut. Let A, B, (*fig.* 165), be the given points, MN a line given in position; it is required to find two points C, D, in it, such, that when the lines AC, AD, BC, BD, are drawn, the angle $CAD = \alpha$, and $CBD = \beta$.



1. From A, B, to MN, draw the lines AE, BF, containing the angles $AED = CAD = \alpha$, $BFD = CBD = \beta$. The lines FE, AE, BF, may then be considered as known, and let ∴ $FE = a$, $AE = b$, $BF = c$. In order to determine the points C, D, put $DE = x$.

2. Now it is easily seen, that $\triangle\, AED$ is similar to $\triangle\, CAD$, and $\triangle\, BFD$ is similar to $\triangle\, CBD$: we ∴ have $CD : AD = AD : DE$, $CD : BD = BD : DF$, and consequently

$$CD = \frac{AD^2}{DE},\ CD = \frac{BD^2}{DF},$$

or, since $DE = x$, $DF = a + x$, $AD^2 = b^2 + x^2 - 2\,bx \times$ $Cos.\,\alpha$, $BD^2 = c^2 + (a+x)^2 - 2c\,(a+x)\,Cos.\,\beta$; therefore

$$CD = \frac{b^2 + x^2 - 2bx\,Cos.\,\alpha}{x}$$

$$CD = \frac{c^2 + (a + x)^2 - 2c\,(a + x)\,Cos.\,\beta}{a + x}.$$

3. If these two expressions for CD are put equal to one another; we then obtain, after the usual reductions, the following equation:

$$(a + 2b\,Cos.\,\alpha - 2c\,Cos.\,\beta)\,x^2 + (a^2 + c^2 - b^2 + 2ab\,Cos.\,\alpha - 2ac\,Cos.\,\beta)\,x = ab^2$$

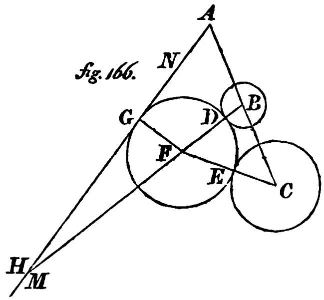
whence x may be determined.

SECTION CXLVI.

Prob. *The position and magnitudes of two circles are given, and also the position of a straight line: describe a circle, which touches these two circles of the straight line.*

Solut. Let B, C, (*fig.* 166) be the centres of the two given circles, F the centre of the required circle, which touches these two circles in D, E, and also the straight line MN given in position in G.

fig. 166.



1. Through the centres B, C, draw a line BC, which meets MN, in A; further, draw the lines FB, FC, FG, the two first of which necessarily pass through the points of contact D, E, (*Euc. III.* 12), and the last is perpendicular to MN. Since the circles, whose centres are B, C, and the line MN have a given position; consequently the angle MAC, the lines AB, BC are given. Put ∴ $MAC = \alpha$, $AB = a$, $BC = b$. Further, let the radius $BD = r$, and the radius

$CE = R$. Now draw the lines BF, CF, and produce the former, till it meets MN in H. If we can now determine the angle CBH, and the radius FD, we can then also find BF: $\therefore$ we have the centre and the radius of the required circle, and consequently the problem is solved. Let $\therefore$ $CBH = \phi$, $FD = x$.

2. From these data we obtain $FB = r + x$, $FC = R + x$, $AHB = \phi - \alpha$. We $\therefore$ have

$$BH = \frac{AB \text{ Sin. } BAH}{\text{Sin. } AHB} = \frac{a \text{ Sin. } \alpha}{\text{Sin. } (\phi - \alpha)},$$

consequently

$$FH = BH - BF = \frac{a \text{ Sin. } \alpha}{\text{Sin. } (\phi - \alpha)} - (r + x).$$

But likewise

$$FH = \frac{GF}{\text{Sin. } AHB} = \frac{x}{\text{Sin. } (\phi - \alpha)};$$

we $\therefore$ have the equation

$$\frac{x}{\text{Sin. } (\phi - \alpha)} = \frac{a \text{ Sin. } \alpha}{\text{Sin. } (\phi - \alpha)} - (r + x),$$

whence we obtain

$$x = \frac{a \text{ Sin. } \alpha - r \text{ Sin. } (\phi - \alpha)}{1 + \text{Sin. } (\phi - \alpha)}.$$

3. Further, in the triangle BFC,

$$CF^2 = BF^2 + BC^2 - 2\, BC \,.\, BF \text{ Cos. } CBF$$

or

$$(x + R)^2 = (x + r)^2 + b^2 - 2\, b\, (x + r) \text{ Cos. } \phi.$$

Hence we obtain

$$x = \frac{r^2 + b^2 - R^2 - 2\, br \text{ Cos. } \phi}{2\, (R - r + b \text{ Cos. } \phi)}.$$

4. If we equate the two expressions found for x in 2, 3, we then obtain, after the usual reduction, the equation

$$2\, b\, (r + a \text{ Sin. } \alpha) \text{ Cos. } \phi + (R^2 - 2\, R\, r + r^2 - b^2) \text{ Sin. } (\phi - \alpha)$$
$$= r^2 + b^2 - R^2 - 2\, a\, (R - r) \text{ Sin. } \alpha.$$

Now $Sin. (\phi - \alpha) = Sin. \phi\ Cos. \alpha - Cos. \phi\ Sin. \alpha$, $R^2 - 2Rr + r^2 - b^2 = (R-r)^2 - b^2 = (R-r+b)\ (R-r-b)$. If these values be substituted in the equation already found, it is transformed into the following one:

$$[2b(r + a\,Sin. \alpha) - (R-r+b)(R-r-b) Sin. \alpha]\, Cos. \phi + (R-r+b)\,(R-r-b)\ Cos. \alpha\ Sin. \phi = r^2 + b^2 - R^2 - 2a\,(R-r)\ Sin. \alpha.$$

5. Divide the equation by $(R-r+b)\ (R-r-b)\ Cos. \alpha$, and put

$$\frac{2b\,(r + a\ Sin. \alpha) - (R-r+b)\ (R-r-b)\ Sin. \alpha}{(R-r+b)\ (R-r-b)\ Cos. \alpha} = Tan. \mu;$$

this gives

$$Tan. \mu\ Cos. \phi + Sin. \phi = \frac{r^2 + b^2 - R^2 - 2a(R-r) Sin. \alpha}{(R-r+b)\ (R-r-b)\ Cos. \alpha}$$

and when both sides of the equation are multiplied by $Cos. \mu$, and $Sin. (\phi + \mu)$ substituted for $Sin. \mu\ Cos. \phi + Cos. \mu\ Sin. \phi$

$$Sin. (\phi + \mu) = \frac{[r^2 + b^2 - R^2 - 2a\,(R-r)\ Sin. \alpha]\ Cos. \mu}{(R-r+b)\,(R-r-b)\ Cos. \alpha}.$$

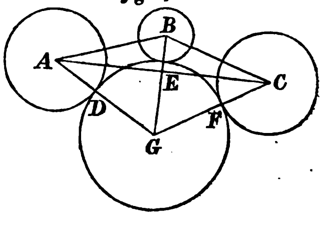
Hence $\phi + \mu$, and consequently also ϕ, may be determined.

SECTION CXLVII.

PROB. *The positions and magnitudes of three circles are given: describe a circle, which touches these circles.*

SOLUT. Let the centres of the given circles be A, B, C (*fig.* 167), and G the centre of the circle, which touches the given ones in D, E, F. With regard to the radii of the given circles, let $AD = R$, $BE = r$, $CF = \rho$; the radius of the required circle $GD = GE = GF = x$. Since the three circles are given in position, consequently the straight lines, which connect

fig. 167.



their centres, and the angle which these lines contain, are likewise given. Let $\therefore$ $AB = a$, $BC = b$, and the angle $ABC = B$.

1. To the centre G of the required circle, draw the lines AG, BG, CG, consequently these pass through the points of contact. If we now put the unknown angle $ABG = \phi$; then from the angles AGB, BGC, we have the two following equations:

$$AG^2 = AB^2 + BG^2 - 2\,AB \,.\, BG\ Cos.\ \phi$$
$$CG^2 = BC^2 + BG^2 - 2\ BC \,.\, BG\ Cos.\ (B - \phi);$$

or, since $AG = R + x$, $BG = r + x$, $CG = \rho + x$,

$$(R + x)^2 = a^2 + (r + x)^2 - 2\,a\,(r + x)\ Cos.\ \phi$$
$$(\rho + x)^2 = b^2 + (r + x)^2 - 2\,b\,(r + x)\ Cos.\ (B - \phi).$$

From the first of these two equations, we get

$$x = \frac{a^2 + r^2 - R^2 - 2\,a\,r\ Cos.\ \phi}{2\,(R - r + a\ Cos.\ \phi)},$$

and from the second,

$$x = \frac{b^2 + r^2 - \rho^2 - 2\,b\,r\ Cos.\ (B - \phi)}{2\,[\rho - r + b\ Cos.\ (B - \phi)]}.$$

2. If the two expressions found for x are equated, we then obtain, after the proper reduction, the following equation:

$a\,(r^2 - 2\,r\,\rho + \rho^2 - b^2)\ Cos.\,\phi - b\,(R^2 - 2\,Rr + r^2 - a^2) \times$
$Cos.\ (B - \phi) = (b^2 + r^2 - \rho^2)\,(R - r) + (a^2 + r^2 - R^2)\,(r - \rho)$

or, since $r^2 - 2\,r\,\rho + \rho^2 - b^2 = (r - \rho)^2 - b^2 = (r - \rho + b) \times (r - \rho - b)$, and $R^2 - 2\,Rr + r^2 - a^2 = (R - r)^2 - a^2 = (R - r + a)\,(R - r - a)$,

$a\,(r - \rho + b)\,(r - \rho - b)\ Cos.\ \phi - b\,(R - r + a)\,(R - r - a)$
$Cos.\,(B - \phi) = (b^2 + r^2 - \rho^2)\,(R - r) + (a^2 + r^2 - R^2)\,(r - \rho).$

Now expand $Cos.\ (B - \phi)$, and divide the whole equation by $b\,(R - r + a)\,(R - r - a)\ Sin.\ B$. This gives, when $Cot.\ B$ is substituted for $\frac{Cos.\ B}{Sin.\ B}$,

$$\left[\frac{a\,(r - \rho + b)\,(r - \rho - b)}{b\,(R - r + a)\,(R - r - a)\,Sin.\,B} - Cot.\ B\right] Cos.\,\phi - Sin.\,\phi$$
$$= \frac{(b^2 + r^2 - \rho^2)\,(R - r) + (a^2 + r^2 - R^2)\,(r - \rho)}{b\,(R - r + a)\,(R - r - a)\ Sin.\ B}.$$

3. Now put

$$\frac{a(r-\rho+b)(r-\rho-b)}{b(R-r+a)(R-r-a)\,Sin.\,B} - Cot.\,B = Tan.\,\mu$$

then μ is a known angle, and we have

$$Tan.\,\mu\,Cos.\,\phi - Sin.\,\phi =$$

$$\frac{(b^2+r^2-\rho^2)(R-r)+(a^2+r^2-R^2)(r-\rho)}{b(R-r+a)(R-r-a)\,Sin.\,B},$$

and when both sides of this equation are multiplied by $Cos.\,\phi$, and $Sin.\,(\mu-\phi)$ substituted for $Sin.\,\mu\,Cos.\,\phi - Cos.\,\mu\,Sin.\,\phi$,

$$Sin.(\mu-\phi)=\frac{[(b^2+r^2-\rho^2)(R-r)+(a^2+r^2-R^2)(r-\rho)]Cos.\mu}{b(R-r+a)(R-r-a)}.$$

Hence $\mu-\phi$, and consequently also ϕ, may be determined.

SECTION CXLVIII.

PROB. *The four sides of a quadrilateral, and its two diagonals, are given: find the segments of these diagonals.*

SOLUT. In the quadrilateral $ABCD$ (*fig.* 168), the four sides AB, BC, CD, AD, and the two diagonals AC, BD, are given; find the segments AE, CE, BE, DE. Let $AB=a$, $BC=b$, $CD=c$, $AD=d$, $AC=f$, $BD=g$, $AE=x$, $BE=y$, and $\therefore CE=f-x$, $DE=g-y$.

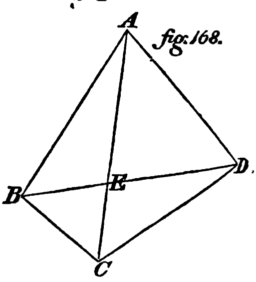


fig. 168.

1. From the four triangles BAD, BCD, ABC, ADC,

$$Cos.\,ABD=\frac{AB^2+BD^2-AD^2}{2AB\,.\,BD}=\frac{a^2+g^2-d^2}{2ag}$$

$$Cos.\,CBD=\frac{BC^2+BD^2-CD^2}{2BC\,.\,BD}=\frac{b^2+g^2-c^2}{2bg}$$

$$Cos.\,ACB = \frac{BC^2 + AC^2 - AB^2}{2BC\,.\,AC} = \frac{b^2 + f^2 - a^2}{2bf}$$

$$Cos.\,ACD = \frac{CD^2 + AC^2 - AD^2}{2CD\,.\,AC} = \frac{c^2 + f^2 - d^2}{2cf}.$$

Further, the three angles AEB, BEC, CED, give the four following equations:

$$AE^2 = AB^2 + BE^2 - 2AB\,.\,BE\ Cos.\,ABD$$
$$CE^2 = BC^2 + BE^2 - 2BC\,.\,BE\ Cos.\,CBD$$
$$BE^2 = BC^2 + CE^2 - 2BC\,.\,CE\ Cos.\,ACB$$
$$DE^2 = CD^2 + CE^2 - 2CD\,.\,CE\ Cos.\,ACD$$

or, when for the lines and cosines their values are substituted, the four following:

$$x^2 = a^2 + y^2 - \frac{a^2 + g^2 - d^2}{g}y$$

$$(f - x)^2 = b^2 + y^2 - \frac{b^2 + g^2 - c^2}{g}y$$

$$y^2 = b^2 + (f - x)^2 - \frac{b^2 + f^2 - a^2}{f}(f - x)$$

$$(g - y)^2 = c^2 + (f - x)^2 - \frac{c^2 + f^2 - d^2}{f}(f - x).$$

2. If the first equation be subtracted from the second, and the third from the fourth, we then obtain the two following equations:

$$f^2 - 2fx = b^2 - a^2 + \frac{a^2 - b^2 + c^2 - d^2}{g}y$$

$$g^2 - 2gy = c^2 - b^2 - \frac{a^2 - b^2 + c^2 - d^2}{f}(f - x);$$

and hence again, by eliminating y,

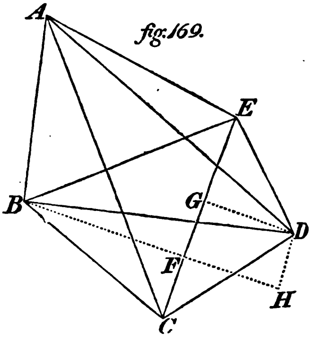
$$x = \frac{f(g^2 + a^2 - d^2)(a^2 - b^2 + c^2 - d^2) - 2fg^2(f^2 + a^2 - b^2)}{(a^2 - b^2 + c^2 - d^2)^2 - 4f^2g^2}.$$

Similar expressions are also obtained for the segments CE, BE, DE.

SECTION CXLIX.

Prob. *From the three angles of a given triangle, a tower is seen, whose base is in the same plane with the triangle; the angles at which it is seen from these, are given: find the distance of the tower from each of these three points.*

Solut. Let AB (*fig.* 169) be the tower, CDE the given triangle; the angles at which the tower is seen from the given points C, D, E, are ACB, ADB, AEB, and let $ACB = \alpha$, $ADB = \beta$, $AEB = \gamma$: find the distances BC, BD, BE.



1. From B, D draw BF, DG perpendicular to CE, and from D draw DH parallel to CE, which meets BF produced in H. Since the triangle CDE is given; consequently the lines CE, CG, DG, are known. Let $\therefore$ $CE = a$, $CG = b$, $DG = FH = c$. If only the lines CF, BF are also determined; we then have likewise the distances sought. Put $\therefore$ $CF = x$, $BF = y$. This gives $BH = y + c$, $DH = GF = b - x$, $EF = a - x$. We consequently have, $BC^2 = BF^2 + CF^2 = y^2 + x^2$, $BD^2 = BH^2 + DH^2 = y^2 + 2cy + c^2 + b^2 - 2bx + x^2$, $BE^2 = BF^2 + EF^2 = y^2 + a^2 - 2ax + x^2$.

2. Since $AB = BC\ Tan.\ \alpha = BD\ Tan.\ \beta = BE\ Tan.\ \gamma$: then also $BC^2\ Tan.\ ^2\alpha = BD^2\ Tan.\ ^2\beta = BE^2\ Tan.\ ^2\gamma$. Substitute now for BC^2, BD^2, BE^2, their values found in 1; hence arise the following equations;

$$Tan.\ ^2\alpha\ (y^2 + x^2)$$
$$= Tan.\ ^2\beta\ (y^2 + x^2 + 2cy - 2bx + b^2 + c^2)$$
$$Tan.\ ^2\alpha\ (y^2 + x^2) = Tan.\ ^2\gamma\ (y^2 + x^2 - 2ax + a^2),$$

or

$$\frac{Tan.\,^2\alpha - Tan.\,^2\beta}{Tan.\,^2\beta}(y^2 + x^2) = 2cy - 2bx + b^2 + c^2$$

$$\frac{Tan.\,^2\alpha - Tan.\,^2\gamma}{Tan.\,^2\gamma}(y^2 + x^2) = -2ax + a^2.$$

3. By trigonometry

$$Tan.\,^2\alpha - Tan.\,^2\beta = \frac{Sin.\,(\alpha + \beta)\,Sin.\,(\alpha - \beta)}{Cos.\,^2\alpha\,Cos.\,^2\beta}$$

$$Tan.\,^2\alpha - Tan.\,^2\gamma = \frac{Sin.\,(\alpha + \gamma)\,Sin.\,(\alpha - \gamma)}{Cos.\,^2\alpha\,Cos.\,^2\gamma};$$

further,

$$Tan.\,^2\beta = \frac{Sin.\,^2\beta}{Cos.\,^2\beta}, \quad Tan.\,^2\gamma = \frac{Sin.\,^2\gamma}{Cos.\,^2\gamma}.$$

4. The substitution of these values in 2, gives the two following equations:

$$\frac{Sin.\,(\alpha+\beta)\,Sin.\,(\alpha-\beta)}{Cos.\,^2\alpha\,Sin.\,^2\beta}(y^2 + x^2) = 2cy - 2bx + b^2 + c^2$$

$$\frac{Sin.\,(\alpha+\gamma)\,Sin.\,(\alpha-\gamma)}{Cos.\,^2\alpha\,Sin.\,^2\gamma}(y^2 + x^2) = -2ax + a^2.$$

5. We eliminate $y^2 + x^2$, by multiplying the first equation by $\frac{Sin.\,(\alpha + \gamma)\,Sin.\,(\alpha - \gamma)}{Sin.\,^2\gamma}$, and the second by $\frac{Sin.\,(\alpha + \beta)\,Sin.\,(\alpha - \beta)}{Sin.\,^2\beta}$, and then subtract them from one another: this gives the following equation:

$$(2cy - 2bx + b^2 + c^2)\frac{Sin.\,(\alpha + \gamma)\,Sin.\,(\alpha - \gamma)}{Sin.\,^2\gamma} -$$

$$(a^2 - 2ax)\frac{Sin.\,(\alpha + \beta)\,Sin.\,(\alpha - \beta)}{Sin.\,^2\beta} = 0.$$

Hence we obtain

$$y = \left[\frac{b}{c} - \frac{a\ Sin.\ ^2\gamma\ Sin.\ (\alpha + \beta)\ Sin.\ (\alpha - \beta)}{c\ Sin.\ ^2\beta\ Sin.\ (\alpha + \gamma)\ Sin.\ (\alpha - \gamma)}\right] x + \frac{a^2\ Sin.\ ^2\gamma\ Sin.\ (\alpha + \beta)\ Sin.\ (\alpha - \beta)}{2\,c\ Sin.\ ^2\beta\ Sin.\ (\alpha + \gamma)\ Sin.\ (\alpha - \gamma)} - \frac{b^2 + c^2}{2c}.$$

If we substitute this value in the second equation in 4, we then obtain a quadratic equation for x, whence x, and consequently also y, may be determined.

6. For the sake of brevity, put

$$\frac{a\ Sin.\ ^2\gamma\ Sin.\ (\alpha + \beta)\ Sin.\ (\alpha - \beta)}{c\ Sin.\ ^2\beta\ Sin.\ (\alpha + \gamma)\ Sin.\ (\alpha - \gamma)} = A,$$

$$\frac{Sin.\ (\alpha + \gamma)\ Sin.\ (\alpha - \gamma)}{Cos.\ ^2\alpha\ Sin.\ ^2\gamma} = B,$$

we consequently have,

$$y = \left[\frac{b}{c} - A\right] x + \tfrac{1}{2}\,a A - \frac{b^2 + c^2}{2\,c}.$$

$$B\,(y^2 + x^2) = -\,2\,ax + a^2.$$

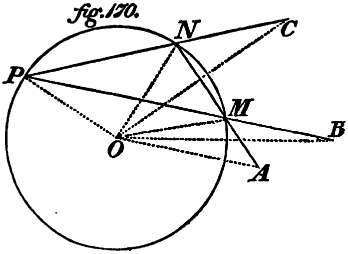
The expressions A, B may be very easily calculated by means of logarithms; the further calculation will be best managed by the given numbers themselves.

SECTION CL.

Prob. *A circle is given in magnitude and position: inscribe a triangle within it, whose sides, or their parts produced, pass through three given points.*

Solut. Let O (*fig.* 170) be the centre of the given circle, MNP the required triangle, whose sides pass through the three given points A, B, C.

Fig. 170.



1. Draw the lines OA, OB, OC: then these, because the points A, B, C, O, are given, are in like manner given in magnitude

and position. Therefore $OA = a$, $OB = b$, $OC = c$, $AOB = \alpha$, $AOC = \beta$.

2. Draw the radii OM, ON, OP; then these lines are radii of the given circle, and consequently given in magnitude, but the position is unknown: $\therefore$ these must be found. Therefore put $AOM = \phi$, $AON = \psi$, $AOP = \zeta$; further, let the radius of the circle $= r$.

3. In the isosceles triangle MON, the angle $MON = \psi - \phi$; consequently $ONM = \dfrac{180^0 - MON}{2} = 90^0 - \frac{1}{2}(\psi - \phi)$. Further, in the triangle AON, the angle $OAN = 180^0 - (ONM + AON) = 90^0 - \frac{1}{2}(\psi + \phi)$.

4. The triangle AON gives the proportion,

$$AO : ON = Sin.\ ONM : Sin.\ OAN,$$

or

$$a : r = Cos.\ \tfrac{1}{2}(\psi - \phi) : Cos.\ \tfrac{1}{2}(\psi + \phi).$$

Hence we obtain the equation

$$a\ Cos.\ \tfrac{1}{2}(\psi + \phi) = r\ Cos.\ \tfrac{1}{2}(\psi - \phi),$$

or, since

$$Cos.\ \tfrac{1}{2}(\phi + \psi) = Cos.\ \tfrac{1}{2}\psi\ Cos.\ \tfrac{1}{2}\phi - Sin.\ \tfrac{1}{2}\psi\ Sin.\ \tfrac{1}{2}\phi,$$
$$Cos.\ \tfrac{1}{2}(\psi - \phi) = Cos.\ \tfrac{1}{2}\psi\ Cos.\ \tfrac{1}{2}\phi + Sin.\ \tfrac{1}{2}\psi\ Sin.\ \tfrac{1}{2}\phi,$$

the following equation

$$(a - r)\ Cos.\ \tfrac{1}{2}\psi\ Cos.\ \tfrac{1}{2}\phi = (a + r)\ Sin.\ \tfrac{1}{2}\psi\ Sin.\ \tfrac{1}{2}\phi.$$

Divide by $(a + r)\ Cos.\ \frac{1}{2}\psi\ Cos.\ \frac{1}{2}\phi$, and substitute $Tan.\ \frac{1}{2}\psi$ for $\dfrac{Sin.\ \frac{1}{2}\psi}{Cos.\ \frac{1}{2}\psi}$, $Tan.\ \frac{1}{2}\phi$ for $\dfrac{Sin.\ \frac{1}{2}\phi}{Cos.\ \frac{1}{2}\phi}$; this gives

$$Tan.\ \tfrac{1}{2}\psi\ Tan.\ \tfrac{1}{2}\phi = \frac{a - r}{a + r}.$$

5. A similar equation is found, when the triangles MOP, BOP are treated in the same way as the triangles MON, AON were in 3, 4. To obtain this, it is only necessary to substitute the line OB for the line OA, the line OP for the line ON, and the angles BOP, BOM, for the angles AON,

AOM, or b for a, $\zeta - \alpha$ for ψ, and $\phi - \alpha$ for ϕ. We then obtain from the equation found in 4, the following one:

$$Tan.\ \tfrac{1}{2}(\zeta - \alpha)\ Tan.\ \tfrac{1}{2}(\phi - \alpha) = \frac{b - r}{b + r}.$$

6. In like manner we find a third equation, when the triangles *NOP*, *COP* are substituted for the triangles *MON*, *AON*, and for this purpose, $OC = c$ is put for $OA = a$, $COP = \zeta - \beta$ for $AON = \psi$, and $CON = \psi - \beta$ for $AOM = \phi$. We consequently have

$$Tan.\ \tfrac{1}{2}(\zeta - \beta)\ Tan.\ \tfrac{1}{2}(\psi - \beta) = \frac{c - r}{c + r}.$$

7. For the sake of abbreviation, put $Tan.\ \tfrac{1}{2}\alpha = m$, $Tan.\ \tfrac{1}{2}\beta = n$, $Tan.\ \tfrac{1}{2}\phi = x$, $Tan.\ \tfrac{1}{2}\psi = y$, $Tan.\ \tfrac{1}{2}\zeta = z$; further $\frac{a - r}{a + r} = A$, $\frac{b - r}{b + r} = B$, $\frac{c - r}{c + r} = C$: then,

$Tan.\ \tfrac{1}{2}(\phi - \alpha) = \frac{x - m}{1 + mx}$, $Tan.\ \tfrac{1}{2}(\zeta - \alpha) = \frac{z - m}{1 + mz}$,

$Tan.\ \tfrac{1}{2}(\psi - \beta) = \frac{y - n}{1 + ny}$, $Tan.\ \tfrac{1}{2}(\zeta - \beta) = \frac{z - n}{1 + nz}$. The three equations in 4, 5, 6 are consequently transformed into the following ones:

$$xy = A$$

$$\frac{x - m}{1 + mx} \times \frac{z - m}{1 + mz} = B$$

$$\frac{y - n}{1 + ny} \times \frac{z - n}{1 + nz} = C.$$

8. From the first of these equations we obtain $y = \frac{A}{x}$, and from the second $z = \frac{B - m^2 + (1 + B)mx}{(1 - Bm^2)x - (1 + B)m}$. If we substitute these values in the third equation, we then obtain, $C =$

$$\frac{A - nx}{nA + x}\left[\frac{B - m^2 + (1 + B)mn + [(1 + B)m - (1 - Bm^2)n]x}{-(1 + B)m + (B - m^2)n + [(1 + B)mn + 1 - Bm^2]x}\right].$$

9. For the sake of brevity, put

$$\begin{aligned} B - m^2 + (1 + B)mn &= M \\ (1+B)m - (1 - Bm^2)n &= N \\ -(1+B)m + (B - m^2)n &= P \\ (1+B)mn + (1 - Bm^2) &= Q. \end{aligned}$$

By these means we obtain the equation

$$\frac{A - nx}{nA + x} \times \frac{M + Nx}{P + Qx} = C,$$

or

$$(CQ + nN)x^2 + (nACQ + CP + nM - NA)x = AM - nACP;$$

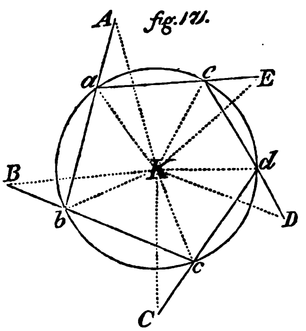
whence x, and consequently also y and z may be determined. Hence further, the angles ϕ, ψ, ζ may be found.

REMARK. This apparently easy problem has long engaged the attention of the greatest geometricians. Cramer first proposed it to Castillon, and this last gave an elegant synthetic proof of it in the Berlin Memoirs of 1776. In the same volume there also appeared another, an Analytical Solution by Lagrange, which I have followed, with the exception of a few alterations in the notation. At Euler's request, Lexell, in the 4th vol. of the Petersburgh Memoirs, gave a geometrical construction of Lagrange's Formula. Altajano, at the age of sixteen, then gave a synthetic solution of this problem, in the 4th vol. of the Memoire della Societa Italiana, and likewise showed, how to describe generally in a given circle a polygon, whose sides pass through any number of given points. In the same volume, there are two more synthetic solutions, one by Malfatti, the other by Giordano. Romano also has a geometrical construction of Lagrange's formula (Nuovo Metodo di applicare alla Sintesi la Soluzione Analytica di qualunque Problema Geometrico. Venezia, 1793), which work I beg to recommend to all those who wish for practice in the geometrical construction of analytical formulæ. Carnot, in his Geometrie de Position, has, as well as Altajano, solved the general problem for the polygon in the following ingenious way.

SECTION CLI.

PROB. *A certain number of points and a circle are given: required to inscribe in this circle a polygon, consisting of as many sides as there are points, so that each side may respectively pass through one of these points.*

SOLUT. Let K (*fig.* 171) be the centre of the given circle,



$abcde$ the required polygon, whose sides pass through the given points A, B, C, D, E.

1. Since the points A, B, C, D, E are given; consequently the lines KA, KB, KC, KD, KE, and the angles AKB, BKC, CKD, DKE, EKA are also given. Let $\therefore KA = a$, $KB = b$, $KC = c$, $KD = d$, $KE = e$, $AKB = \alpha$, $BKC = \beta$, $CKD = \gamma$, $DKE = \delta$, $EKA = \epsilon$; further, let the radius of the circle $= r$. Now if the angles AKa, BKb, CKc, DKd, EKe are known; then the points a, b, c, d, e, are determined, and consequently the problem is solved. Put $\therefore AKa = \tau$, $BKb = v$, $CKc = \phi$, $DKd = \chi$, $EKe = \psi$.

2. Then in the triangle AKb, we have

$$AK + bK : AB - bK =$$
$$Tan. \tfrac{1}{2}(AbK + bAK) : Tan. \tfrac{1}{2}(AbK - bAK).$$

Now $AbK + bAK = 180^\circ - AKb = 180^\circ - (\alpha + v)$, $AbK - bAK = baK - bAK = AKa = \tau$; consequently $Tan. \frac{1}{2}(AbK + bAK) = Tan. [90^\circ - \frac{1}{2}(\alpha + v)] = Cot. \frac{1}{2}(\alpha + v) = \dfrac{1}{Tan. \frac{1}{2}(\alpha + v)}$, $Tan. \frac{1}{2}(AbK - bAK) = Tan. \frac{1}{2}\tau$. We $\therefore$ have

$$a + r : a - r = \frac{1}{Tan. \frac{1}{2}(\alpha + v)} : Tan. \tfrac{1}{2}\tau,$$

or,

$$\frac{a - r}{a + r} = Tan. \tfrac{1}{2}\tau \, Tan. \tfrac{1}{2}(\alpha + v),$$

or,

$$\frac{a - r}{a + r} = Tan. \tfrac{1}{2}\tau \frac{Tan. \frac{1}{2}\alpha + Tan. \frac{1}{2}v}{1 - Tan. \frac{1}{2}\alpha \, Tan. \frac{1}{2}v};$$

whence we obtain,

$$Tan. \tfrac{1}{2}\tau = \frac{\dfrac{a - r}{a + r} - \dfrac{a - r}{a + r} Tan. \frac{1}{2}\alpha \, Tan. \frac{1}{2}v}{Tan. \frac{1}{2}\alpha + Tan. \frac{1}{2}v}$$

3. A similar equation is found for each side of the polygon. Thus we obtain as many equations as the polygon has sides, and these equations are:

$$Tan.\, \tfrac{1}{2}\, \tau = \frac{\dfrac{a-r}{a+r} - \dfrac{a-r}{a+r}\, Tan.\, \tfrac{1}{2}\, \alpha\; Tan.\, \tfrac{1}{2}\, \upsilon}{Tan.\, \tfrac{1}{2}\, \alpha + Tan.\, \tfrac{1}{2}\, \upsilon}$$

$$Tan.\, \tfrac{1}{2}\, \upsilon = \frac{\dfrac{b-r}{b+r} - \dfrac{b-r}{b+r}\, Tan.\, \tfrac{1}{2}\, \beta\; Tan.\, \tfrac{1}{2}\, \phi}{Tan.\, \tfrac{1}{2}\, \beta + Tan.\, \tfrac{1}{2}\, \phi}$$

$$Tan.\, \tfrac{1}{2}\, \phi = \frac{\dfrac{c-r}{c+r} - \dfrac{c-r}{c+r}\, Tan.\, \tfrac{1}{2}\, \gamma\; Tan.\, \tfrac{1}{2}\, \chi}{Tan.\, \tfrac{1}{2}\, \gamma + Tan.\, \tfrac{1}{2}\, \chi}$$

$$Tan.\, \tfrac{1}{2}\, \chi = \frac{\dfrac{d-r}{d+r} - \dfrac{d-r}{d+r}\, Tan.\, \tfrac{1}{2}\, \delta\; Tan.\, \tfrac{1}{2}\, \psi}{Tan.\, \tfrac{1}{2}\, \delta + Tan.\, \tfrac{1}{2}\, \psi}$$

$$Tan.\, \tfrac{1}{2}\, \psi = \frac{\dfrac{e-r}{e+r} - \dfrac{e-r}{e+r}\, Tan.\, \tfrac{1}{2}\, \varepsilon\; Tan.\, \tfrac{1}{2}\, \tau}{Tan.\, \tfrac{1}{2}\, \varepsilon + Tan.\, \tfrac{1}{2}\, \tau}$$

4. By means of these equations, the number of which is always equal to the number of the unknown magnitudes, it is easy to determine these last. If, for instance, we wish to find ϕ: it is merely necessary to substitute in the equation for χ its value taken from the fourth, by which we obtain an equation between ϕ and ψ. From this again we then obtain an equation between ϕ and τ, by substituting for ψ its value taken from the fourth. From this again we get an equation between ϕ and υ, by substituting for τ its value in terms of υ taken from the first equation, and lastly, we obtain an equation, which only contains ϕ, by substituting for υ its value in terms of ϕ taken from the second equation. The solution of this last equation then gives the value of ϕ.

5. For the sake of brevity, put $Tan.\, \frac{1}{2}\, \tau = \tau'$, $Tan.\, \frac{1}{2}\, \upsilon = \upsilon'$, $Tan.\, \frac{1}{2}\, \phi = \phi'$, $Tan.\, \frac{1}{2}\, \chi = \chi'$, $Tan.\, \frac{1}{2}\, \psi = \psi'$; by these means the equations already found take the following form:

35. $Tan.^2\phi - Tan.^2\psi = \dfrac{Sin.(\phi+\psi)\,Sin.(\phi-\psi)}{Cos.^2\phi\,Cos.^2\psi}$

36. $Cot.^2\phi - Cot.^2\psi = \dfrac{Sin.(\phi+\psi)\,Sin.(\psi-\phi)}{Sin.^2\phi\,Sin.^2\psi}$

37. $\dfrac{Sin.\,\phi + Sin\,\psi}{Sin.\,\phi - Sin.\,\psi} = \dfrac{Tan.\,\frac{1}{2}(\phi+\psi)}{Tan.\,\frac{1}{2}(\phi-\psi)}$

38. $\dfrac{Sin.\,\phi + Sin.\,\psi}{Cos.\,\phi + Cos.\,\psi} = Tan.\,\frac{1}{2}(\phi+\psi)$

39. $\dfrac{Sin.\,\phi + Sin.\,\psi}{Cos.\,\phi - Cos.\,\psi} = Cot.\,\frac{1}{2}(\psi-\phi)$

40. $\dfrac{Sin.\,\phi - Sin.\,\psi}{Cos.\,\phi + Cos.\,\psi} = Tan.\,\frac{1}{2}(\phi-\psi)$

41. $\dfrac{Sin.\,\phi - Sin.\,\psi}{Cos.\,\phi - Cos.\,\psi} = -\,Cot.\,\frac{1}{2}(\phi+\psi)$

42. $\dfrac{Cos.\,\phi + Cos.\,\psi}{Cos.\,\phi - Cos.\,\psi} = \dfrac{Cot.\,\frac{1}{2}(\phi+\psi)}{Tan.\,\frac{1}{2}(\psi-\phi)}$

43. $\dfrac{Tan.\,\phi + Tan.\,\psi}{Tan.\,\phi - Tan.\,\psi} = \dfrac{Sin.(\phi+\psi)}{Sin.(\phi-\psi)}$

44. $\dfrac{Cot.\,\phi + Cot.\,\psi}{Cot.\,\phi - Cot.\,\psi} = \dfrac{Sin.(\phi+\psi)}{Sin.(\psi-\phi)}$

45. $\dfrac{Sin\,\phi}{1 + Cos.\,\phi} = Tan.\,\frac{1}{2}\phi$

46. $\dfrac{Sin.\,\phi}{1 - Cos.\,\phi} = Cot.\,\frac{1}{2}\phi$

47. $\dfrac{1 - Cos.\,\phi}{1 + Cos.\,\phi} = Tan.^2\,\frac{1}{2}\phi$

FINIS.

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$$10.\ Sin.\,(\phi \pm \psi) = Sin.\,\phi\ Cos.\,\psi \pm Cos.\,\phi\ Sin.\,\psi$$

$$11.\ Cos.\,(\phi \pm \psi) = Cos.\,\phi\ Cos.\,\psi \mp Sin.\,\phi\ Sin.\,\psi$$

$$12.\ Sin.\,\phi\ Cos.\,\psi = \tfrac{1}{2}\,[Sin.\,(\phi + \psi) + Sin.\,(\phi - \psi)]$$

$$13.\ Cos.\,\phi\ Sin.\,\psi = \tfrac{1}{2}\,[Sin.\,(\phi + \psi) - Sin.\,(\phi - \psi)]$$

$$14.\ Cos.\,\phi\ Cos.\,\psi = \tfrac{1}{2}\ Cos.\,(\phi - \psi) + Cos.\,(\phi + \psi)]$$

$$15.\ Sin.\,\phi\ Sin.\,\psi = \tfrac{1}{2}\,[Cos.\,(\phi - \psi) - Cos.\,(\phi + \psi)]$$

$$16.\ Sin.\,\phi + Sin.\,\psi = 2\,Sin.\,\tfrac{1}{2}\,(\phi + \psi)\ Cos.\,\tfrac{1}{2}\,(\phi - \psi)$$

$$17.\ Sin.\,\phi - Sin.\,\psi = 2\,Cos.\,\tfrac{1}{2}\,(\phi + \psi)\ Sin.\,\tfrac{1}{2}\,(\phi - \psi)$$

$$18.\ Cos.\,\phi + Cos.\,\psi = 2\,Cos.\,\tfrac{1}{2}\,(\phi + \psi)\ \overset{Cos}{\cancel{Sin.}}\,\tfrac{1}{2}\,(\phi - \psi)$$

$$19.\ Cos.\,\phi - Cos.\,\psi = -\,2\,Sin.\,\tfrac{1}{2}\,(\phi + \psi)\ Sin.\,\tfrac{1}{2}\,(\phi - \psi)$$

$$20.\ Tan.\,(\phi \pm \psi) = \frac{Tan.\,\phi \pm Tan.\,\psi}{1 \mp Tan.\,\phi\ Tan.\,\psi}$$

$$21.\ Cot.\,(\phi \pm \psi) = \frac{Cot.\,\phi\ Cot.\,\psi \pm 1}{Cot.\,\psi \mp Cot.\,\phi}$$

$$22.\ Tan.\,\phi + Tan.\,\psi = \frac{Sin.\,(\phi + \psi)}{Cos.\,\phi\ Cos.\,\psi}$$

$$23.\ Tan.\,\phi - Tan.\,\psi = \frac{Sin.\,(\phi - \psi)}{Cos.\,\phi\ Cos.\,\psi}$$

$$24.\ Cot.\,\phi + Cot.\,\psi = \frac{Sin.\,(\phi + \psi)}{Sin.\,\phi\ Sin.\,\psi}$$

$$25.\ Cot.\,\phi - Cot.\,\psi = \frac{Sin.\,(\psi - \phi)}{Sin.\,\phi\ Sin.\,\psi}$$

$$26.\ Sin.\,2\,\phi = 2\,Sin.\,\phi\ Cos.\,\phi$$

$$27.\ Cos.\,2\,\phi = Cos.^2\,\phi - Sin.^2\,\phi$$

$$28.\ Sin.^2\,\phi = \tfrac{1}{2}\,(1 - Cos.\,2\,\phi)$$

$$29.\ Cos.^2\,\phi = \tfrac{1}{2}\,(1 + Cos.\,2\,\phi)$$

$$30.\ Sec.^2\,\phi = 1 + Tan.^2\,\phi$$

$$31.\ Cosec.^2\,\phi = 1 + Cot.^2\,\phi$$

$$32.\ Sin.^2\,\phi - Sin.^2\,\psi = Sin.\,(\phi + \psi)\ Sin.\,(\phi - \psi)$$

$$33.\ Cos.^2\,\phi - Cos.^2\,\psi = -\,Sin.\,(\phi + \psi)\ Sin.\,(\phi - \psi)$$

$$34.\ Cos.^2\,\phi - Sin.^2\,\psi = Cos.\,(\phi + \psi)\ Cos.\,(\phi - \psi)$$

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